

# Math 211

Lecture #24

Linear Systems of ODEs

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## General Linear Systems

$$x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1$$

$$x_2' = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2$$

$$\vdots = \quad \vdots$$

$$x_n' = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n$$

- The coefficients can depend on  $t$ .

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- Set

$$\mathbf{x} = (x_1, x_2, \dots, x_n)^T$$

$$\mathbf{f}(t) = (f_1(t), f_2(t), \dots, f_n(t))^T$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- The system becomes  $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ .

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## Homogeneous Systems

An *homogeneous* system is one of the form

$$\mathbf{x}' = A\mathbf{x}$$

**Proposition:** Suppose that  $\mathbf{x}_1(t)$ ,  $\mathbf{x}_2(t)$ ,  $\dots$ , and  $\mathbf{x}_k(t)$  are solutions to the homogeneous system  $\mathbf{x}' = A\mathbf{x}$ , and  $c_1, c_2, \dots$ , and  $c_k$  are scalars. Then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \dots + c_k\mathbf{x}_k(t)$$

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.

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## Very Important Example

- The system

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

has solutions

$$\mathbf{x}_1(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

- Verify by direct substitution.
- Proposition  $\Rightarrow \mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$  is a solution for any constants  $C_1$  and  $C_2$ .
- Is this the general solution?

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- Let  $\mathbf{y}$  be a solution of  $\mathbf{y}' = A\mathbf{y}$ . Can we find  $C_1$  and  $C_2$  so that

$$\mathbf{y}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \quad \text{for all } t?$$

- Let's ask a simpler question. Can we find  $C_1$  and  $C_2$  so that

$$\mathbf{y}(0) = C_1\mathbf{x}_1(0) + C_2\mathbf{x}_2(0)?$$

- Yes, since  $\mathbf{x}_1(0)$  and  $\mathbf{x}_2(0)$  are linearly independent.
- Uniqueness theorem  $\Rightarrow$

$$\mathbf{y}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \quad \text{for all } t.$$

- Thus, every solution to  $\mathbf{x}' = A\mathbf{x}$  is a linear combination of  $\mathbf{x}_1$  and  $\mathbf{x}_2$ .
- Can we generalize this result?

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[Structure](#)

### Key Point in the Argument

- Need to solve the equation

$$y_0 = C_1x_1(0) + C_2x_2(0)$$

for any  $y_0 = y(0)$ .

- Possible if  $x_1(0)$  and  $x_2(0)$  are linearly independent.
- Uniqueness then implies that

$$y(t) = C_1x_1(t) + C_2x_2(t) \quad \text{for all } t.$$

- We needed  $x_1(t)$  and  $x_2(t)$  to be linearly independent at only one point.

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**Proposition:**  $x_1(t)$ ,  $x_2(t)$ ,  $\dots$ , and  $x_k(t)$  solutions to the homogeneous system  $x' = Ax$  on the interval  $I$ .

1. If  $x_1(t_0)$ ,  $x_2(t_0)$ ,  $\dots$ , and  $x_k(t_0)$  are linearly independent for some  $t_0 \in I$ , then they are linearly independent for all  $t \in I$ .
2. If  $x_1(t_0)$ ,  $x_2(t_0)$ ,  $\dots$ , and  $x_k(t_0)$  are linearly dependent for some  $t_0 \in I$ , then they are linearly dependent for all  $t \in I$ .

**Definition:** A set of  $k$  solutions to the linear system  $x' = Ax$  is *linearly independent* if they are linearly independent at one value of  $t$ .

- Proposition  $\Rightarrow$  the solutions are linearly independent for all values of  $t$ .

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[Example](#)

### Structure of the Solution Space

**Theorem:** Suppose that  $x_1(t)$ ,  $x_2(t)$ ,  $\dots$ , and  $x_n(t)$  are linearly independent solutions to the  $n \times n$  homogeneous system  $x' = Ax$  on the interval  $I$ . Then every solution is a linear combination of  $x_1(t)$ ,  $x_2(t)$ ,  $\dots$ , and  $x_n(t)$ .

- That is, if  $x(t)$  is any solution, then there are constants  $C_1$ ,  $C_2$ ,  $\dots$ , and  $C_n$  such that

$$x(t) = C_1x_1(t) + C_2x_2(t) + \dots + C_nx_n(t).$$

- The general solution is a linear combination of  $x_1(t)$ ,  $x_2(t)$ ,  $\dots$ , and  $x_n(t)$ .

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[Very Important Example](#)

### Solution Strategy

- The obvious strategy for completely solving an  $n \times n$  homogeneous system is to look for  $n$  linearly independent solutions.

**Definition:** A set of  $n$  linear independent solutions to the  $n \times n$  homogeneous system  $\mathbf{x}' = A\mathbf{x}$  is called a *fundamental set of solutions*.

- We will develop methods of finding fundamental sets of solutions.

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### Examples: $\mathbf{x}' = A\mathbf{x}$

- Example 1:  $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\mathbf{x}_1(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

is a fundamental set of solutions.

- Example 2:  $A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

is a fundamental set of solutions.

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[Linear independence](#)

### Linear Systems with Constant Coefficients

- We will solve homogeneous systems,  $\mathbf{x}' = A\mathbf{x}$ , first.
- Consider the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{x}_0.$$

- To motivate what we do, we will start with the easiest case, dimension = 1. The initial value problem is

$$x' = ax \quad \text{with} \quad x(0) = x_0.$$

- One equation:  $x' = ax$ , where  $a$  is a constant. The solution is

$$x(t) = x_0 e^{at} = e^{ta} x_0.$$

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### Direct Analogy

- The direct analogy in dimension  $n$  would be

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0.$$

- Can we define  $e^{tA}$ ?
- Will this formula be a solution to the initial value problem?
- Can we compute  $e^{tA}\mathbf{x}_0$ ?

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### Exponential of a Matrix

**Definition:** The *exponential* of the  $n \times n$  matrix  $A$  is the  $n \times n$  matrix

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots \\ &= \sum_0^{\infty} \frac{1}{n!}A^n. \end{aligned}$$

- Examples:

- ♦  $A = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \Rightarrow e^A = \begin{pmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{pmatrix}.$
- ♦  $e^{\lambda I} = e^{\lambda}I, \quad e^{0I} = I.$

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### A Very Important Fact

- If  $A$  is an  $n \times n$  matrix, then

$$\frac{d}{dt}e^{tA} = Ae^{tA}.$$

**Theorem:** The solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}$$

is given by  $\mathbf{x}(t) = e^{tA}\mathbf{v}.$

- We have answered two of the three questions.
  - ♦ However computing  $e^{tA}$  is not easy.

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Analogy

Exponential