# Math 211

Lecture #24

Linear Systems of ODEs

October 22, 2003

# General Linear Systems

$$
\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1 \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2 \\
\vdots &= \qquad \vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n
\end{aligned}
$$

- The coefficients can depend on $t$.

- Set

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$$

$$\mathbf{f}(t) = (f_1(t), f_2(t), \ldots, f_n(t))^T$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- The system becomes $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$.

# Homogeneous Systems

An *homogeneous* system is one of the form

$$\mathbf{x}' = A\mathbf{x}$$

**Proposition:** Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_k(t)$ are solutions to the homogeneous system $\mathbf{x}' = A\mathbf{x}$, and $c_1$, $c_2$, . . . , and $c_k$ are scalars. Then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_k\mathbf{x}_k(t)$$

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.

# Very Important Example

- The system

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

has solutions

$$\mathbf{x}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

  - Verify by direct substitution.

- Proposition $\Rightarrow \mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$ is a solution for any constants $C_1$ and $C_2$.

  - Is this the general solution?

- Let $\mathbf{y}$ be a solution of $\mathbf{y}' = A\mathbf{y}$. Can we find $C_1$ and $C_2$ so that

$$\mathbf{y}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \quad \text{for all } t?$$

- Let's ask a simpler question. Can we find $C_1$ and $C_2$ so that

$$\mathbf{y}(0) = C_1\mathbf{x}_1(0) + C_2\mathbf{x}_2(0)?$$

  - Yes, since $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ are linearly independent.

- Uniqueness theorem $\Rightarrow$

$$\mathbf{y}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \quad \text{for all } t.$$

- Thus, every solution to $\mathbf{x}' = A\mathbf{x}$ is a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$.
- Can we generalize this result?

# Key Point in the Argument

- Need to solve the equation

$$\mathbf{y}_0 = C_1\mathbf{x}_1(0) + C_2\mathbf{x}_2(0)$$

  for any $\mathbf{y}_0 = \mathbf{y}(0)$.

- Possible if $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ are linearly independent.
- Uniqueness then implies that

$$\mathbf{y}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \quad \text{for all } t \quad .$$

- We needed $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ to be linearly independent at only one point.

**Proposition:** $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, ..., and $\mathbf{x}_k(t)$ solutions to the homogeneous system $\mathbf{x}' = A\mathbf{x}$ on the interval $I$.

1. If $\mathbf{x}_1(t_0)$, $\mathbf{x}_2(t_0)$, ..., and $\mathbf{x}_k(t_0)$ are linearly independent for some $t_0 \in I$, then they are linearly independent for all $t \in I$.
2. If $\mathbf{x}_1(t_0)$, $\mathbf{x}_2(t_0)$, ..., and $\mathbf{x}_k(t_0)$ are linearly dependent for some $t_0 \in I$, then they are linearly dependent for all $t \in I$.

**Definition:** A set of $k$ solutions to the linear system $\mathbf{x}' = A\mathbf{x}$ is *linearly independent* if they are linearly independent at one value of $t$.

- Proposition $\Rightarrow$ the solutions are linearly independent for all values of $t$.

# Structure of the Solution Space

**Theorem:** Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_n(t)$ are linearly independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ on the interval $I$. Then every solution is a linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_n(t)$.

- That is, if $\mathbf{x}(t)$ is any solution, then there are constants $C_1$, $C_2$, . . . , and $C_n$ such that

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) + \cdots + C_n\mathbf{x}_n(t).$$

- The general solution is a linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_n(t)$.

# Solution Strategy

- The obvious strategy for completely solving an $n \times n$ homogeneous system is to look for $n$ linearly independent solutions.

**Definition:** A set of $n$ linear independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ is called a *fundamental set of solutions.*

- We will develop methods of finding fundamental sets of solutions.

# Examples: $\mathbf{x}' = A\mathbf{x}$

- Example 1: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$\mathbf{x}_1(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

is a fundamental set of solutions.

- Example 2: $A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

is a fundamental set of solutions.

# Linear Systems with Constant Coefficients

- We will solve homogeneous systems, $\mathbf{x}' = A\mathbf{x}$, first.
- Consider the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{x}_0.$$

- To motivate what we do, we will start with the easiest case, dimension $= 1$. The initial value problem is

$$x' = ax \quad \text{with} \quad x(0) = x_0.$$

  - One equation: $x' = ax$, where $a$ is a constant. The solution is

$$x(t) = x_0 e^{at} = e^{ta} x_0.$$

# Direct Analogy

- The direct analogy in dimension $n$ would be

$$\mathbf{x}(t) = e^{tA}\mathbf{x}_0.$$

- Can we define $e^{tA}$?
- Will this formula be a solution to the initial value problem?
- Can we compute $e^{tA}\mathbf{x}_0$?

# Exponential of a Matrix

**Definition:** The *exponential* of the $n \times n$ matrix $A$ is the $n \times n$ matrix

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$
$$= \sum_0^\infty \frac{1}{n!}A^n.$$

- Examples:
  - $A = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \Rightarrow e^A = \begin{pmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{pmatrix}.$
  - $e^{\lambda I} = e^\lambda I. \quad e^{0I} = I.$

# A Very Important Fact

- If $A$ is an $n \times n$ matrix, then

$$\frac{d}{dt} e^{tA} = A e^{tA}.$$

**Theorem:** The solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}$$

is given by $\mathbf{x}(t) = e^{tA}\mathbf{v}$.

- We have answered two of the three questions.
  - ♦ However computing $e^{tA}$ is not easy.