

Math 211

Lecture #25

The Exponential of a Matrix

October 24, 2003

Homogeneous Systems

- These are systems of the form

$$\mathbf{x}' = A\mathbf{x},$$

where A is an $n \times n$ matrix.

- We are looking primarily at homogeneous systems with constant coefficients.

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Structure of the Solution Space

Theorem: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, \dots , and $\mathbf{x}_n(t)$ are linearly independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ on the interval I . Then every solution is a linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, \dots , and $\mathbf{x}_n(t)$.

- That is, if $\mathbf{x}(t)$ is a solution, then there are constants C_1 , C_2 , \dots , and C_n such that

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) + \dots + C_n\mathbf{x}_n(t).$$

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Solution Strategy

- The obvious strategy for completely solving the system is to look for n linearly independent solutions.

Definition: A set of n linear independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ is called a *fundamental set of solutions*.

- We will look for fundamental sets of solutions.

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Exponential of a Matrix

Definition: The *exponential* of the $n \times n$ matrix A is the $n \times n$ matrix

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots \\ &= \sum_0^{\infty} \frac{1}{n!}A^n. \end{aligned}$$

- Example: $e^{\lambda I} = e^{\lambda}I$.

Theorem: The solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}$$

is given by $\mathbf{x}(t) = e^{tA}\mathbf{v}$.

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Truncation

- If \mathbf{v} is a vector, we have

$$e^{tA}\mathbf{v} = \mathbf{v} + tA\mathbf{v} + \frac{t^2}{2!}A^2\mathbf{v} + \frac{t^3}{3!}A^3\mathbf{v} + \dots$$

- If $A\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = \mathbf{v}$.
- If $A^2\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = \mathbf{v} + tA\mathbf{v}$.
- If $A^k\mathbf{v} = \mathbf{0}$, then

$$e^{tA}\mathbf{v} = \mathbf{v} + tA\mathbf{v} + \frac{t^2}{2!}A^2\mathbf{v} + \dots + \frac{t^{k-1}}{(k-1)!}A^{k-1}\mathbf{v}.$$

- We can compute $e^{tA}\mathbf{v}$ if \mathbf{v} is in the nullspace of A or of a power of A .

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Example 1

Consider

$$A = \begin{pmatrix} -6 & 4 \\ -12 & 8 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

- $A\mathbf{v} = \mathbf{0}$. The series for $e^{tA}\mathbf{v}$ truncates

$$\begin{aligned} e^{tA}\mathbf{v} &= I\mathbf{v} + tA\mathbf{v} + \cdots \\ &= \mathbf{v}. \end{aligned}$$

- Although we cannot compute e^{tA} , we can compute $e^{tA}\mathbf{v}$.

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The Law of Exponents

Proposition: Suppose that A and B are $n \times n$ matrices.

1. $e^{A+B} = e^A e^B \Leftrightarrow AB = BA$ (A and B commute).
2. The exponential e^A is invertible and its inverse is e^{-A} .

Suppose that A an $n \times n$ matrix, and λ a number

- Then $A = \lambda I + [A - \lambda I]$, and λI & $A - \lambda I$ commute.
- Therefore $e^{tA} = e^{\lambda t I} e^{t[A - \lambda I]} = e^{\lambda t} e^{t[A - \lambda I]}$.
 - ♦ This simple identity will enable us to compute the exponentials we need. We will link this identity with previous examples.

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$$e^{\lambda I} = e^{\lambda I}$$

Proposition

Proposition: Suppose that A is an $n \times n$ matrix, λ is a number, and \mathbf{v} is a vector.

1. If $[A - \lambda I]\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$.
2. If $[A - \lambda I]^2\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}(\mathbf{v} + t[A - \lambda I]\mathbf{v})$.
3. If $[A - \lambda I]^k\mathbf{v} = \mathbf{0}$, then

$$\begin{aligned} e^{tA}\mathbf{v} &= e^{\lambda t} \left(\mathbf{v} + t[A - \lambda I]\mathbf{v} + \frac{t^2}{2!}[A - \lambda I]^2\mathbf{v} + \cdots \right. \\ &\quad \left. + \frac{t^{k-1}}{(k-1)!}[A - \lambda I]^{k-1}\mathbf{v} \right). \end{aligned}$$

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Truncation

Law of Exponents

Example 2

Consider

$$A = \begin{pmatrix} 7 & 4 \\ -8 & -5 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

- $A\mathbf{v}_1 = -\mathbf{v}_1$ so

$$\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^{-t}\mathbf{v}_1 = e^{-t} \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

- $A\mathbf{v}_2 = 3\mathbf{v}_2$ so

$$\mathbf{x}_2(t) = e^{tA}\mathbf{v}_2 = e^{3t}\mathbf{v}_2 = e^{3t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

- Since \mathbf{v}_1 and \mathbf{v}_2 are linearly independent, \mathbf{x}_1 and \mathbf{x}_2 form a fundamental set of solutions to $\mathbf{x}' = A\mathbf{x}$.

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The Impact of the Proposition

- We can compute $e^{tA}\mathbf{v}$ whenever \mathbf{v} is in the nullspace of $A - \lambda I$ or of some power of $A - \lambda I$ for some number λ .
- Part 1 of the Proposition was all we needed in the example.
- Part 1 of the Proposition is all we will need for most matrices.
- We will use the rest of the Proposition to handle exceptional cases.
- Notice that

$$[A - \lambda I]\mathbf{v} = \mathbf{0} \quad \Leftrightarrow \quad A\mathbf{v} = \lambda\mathbf{v}.$$

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Eigenvalues & Eigenvectors

Definition: λ is an *eigenvalue* of A if there is a nonzero vector \mathbf{v} such that $A\mathbf{v} = \lambda\mathbf{v}$. If λ is an eigenvalue of A , then any vector \mathbf{v} such that $A\mathbf{v} = \lambda\mathbf{v}$ is called an *eigenvector associated with* λ .

- If λ an eigenvalue of A , and \mathbf{v} is an associated nonzero eigenvector, then $\mathbf{x}(t) = e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$ is a solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}.$$

- Solutions of the form $e^{\lambda t}\mathbf{v}$ are called *exponential solutions*.
- How do we find eigenvalues and eigenvectors?

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IVP

Finding Eigenvalues

λ is an eigenvalue of A

- \Leftrightarrow there is a vector $\mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.
- \Leftrightarrow there is a vector $\mathbf{v} \neq \mathbf{0}$ such that $[A - \lambda I]\mathbf{v} = \mathbf{0}$
- $\Leftrightarrow A - \lambda I$ has a nontrivial nullspace.
- $\Leftrightarrow A - \lambda I$ is singular.
- $\Leftrightarrow \det(A - \lambda I) = 0$.

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Example part 1

$$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -4 - \lambda & 2 \\ -3 & 1 - \lambda \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (-4 - \lambda)(1 - \lambda) + 6 \\ &= \lambda^2 + 3\lambda + 2 \\ &= (\lambda + 1)(\lambda + 2) \end{aligned}$$

- A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

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[Finding eigenvalues](#)

The Characteristic Polynomial

- If A is an $n \times n$ matrix $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree n .

Definition: The *characteristic polynomial* of the $n \times n$ matrix A is

$$p(\lambda) = \det(A - \lambda I).$$

The *characteristic equation* is $p(\lambda) = 0$.

- Thus, the eigenvalues of A are the roots of the characteristic equation.

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Finding Eigenvectors

\mathbf{v} is an eigenvector associated with the eigenvalue λ if

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ \Leftrightarrow (A - \lambda I)\mathbf{v} &= \mathbf{0} \\ \Leftrightarrow \mathbf{v} &\in \text{null}(A - \lambda I) \end{aligned}$$

- The set of all eigenvectors associated to the eigenvalue λ is equal to the nullspace of $A - \lambda I$.
 - ♦ It is a subspace of \mathbb{R}^n .
 - ♦ It is called the *eigenspace* of λ .

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Finding eigenvalues

Example part 2

The matrix $A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

- $\lambda_1 = -1$: $A - \lambda_1 I = \begin{pmatrix} -4+1 & 2 \\ -3 & 1+1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix}$
 - ♦ $\mathbf{v}_1 = (2, 3)^T$ is an eigenvector.
 - ♦ $\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^{\lambda_1 t}\mathbf{v}_1 = e^{-t}(2, 3)^T$ is a solution.
- $\lambda_2 = -2$: $A - \lambda_2 I = \begin{pmatrix} -4+2 & 2 \\ -3 & 1+2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix}$
 - ♦ $\mathbf{v}_2 = (1, 1)^T$ is an eigenvector.
 - ♦ $\mathbf{x}_2(t) = e^{tA}\mathbf{v}_2 = e^{\lambda_2 t}\mathbf{v}_2 = e^{-2t}(1, 1)^T$ is a solution.

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Finding eigenvalues

Example part 3

- $\mathbf{x}_1(0) = \mathbf{v}_1$ and $\mathbf{x}_2(0) = \mathbf{v}_2$ are linearly independent.
- \mathbf{x}_1 and \mathbf{x}_2 form a fundamental set of solutions.
- The general solution is the set of all linear combinations:

$$\begin{aligned} \mathbf{x}(t) &= C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \\ &= C_1e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2C_1e^{-t} + C_2e^{-2t} \\ 3C_1e^{-t} + C_2e^{-2t} \end{pmatrix} \end{aligned}$$

Our Solution Strategy for $\mathbf{x}' = A\mathbf{x}$

If A is $n \times n$, we are looking for n linearly independent solutions.

- Each eigenvalue λ of A has by definition an associated nonzero eigenvector \mathbf{v} . This gives us the solution, $\mathbf{x}(t) = e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$.
- The eigenvalues of A are the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I) = 0$.
 - ♦ $p(\lambda)$ has degree n , and usually has n roots.
- Therefore, there are usually n different solutions.
 - ♦ Are they linearly independent?

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[Example](#)

Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A , which are the roots of $\det(A - \lambda I) = 0$.
- For each eigenvalue λ find the eigenspace, which is equal to $\text{null}(A - \lambda I)$.
- If λ is an eigenvalue and \mathbf{v} is an associated nonzero eigenvector, $\mathbf{x}(t) = e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$ is a solution.
- Show that n of these are linearly independent, *if we can*.
 - ♦ This must be explored further.

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[Solution strategy](#)

Solving $\mathbf{x}' = A\mathbf{x}$

Cases to be Considered

- Distinct real eigenvalues.
 - ♦ In this case the method works as described.
- Complex eigenvalues.
 - ♦ The method yields complex solutions, but we will want real solutions.
- Repeated eigenvalues.
 - ♦ The method does not always give enough solutions.
 - ♦ This is the hard case, but we have not used all of the Proposition.

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Linear Independence

Definition: A set of k solutions to the linear system $\mathbf{x}' = A\mathbf{x}$ is *linearly independent* if they are linearly independent at one value of t .

- Proposition \Rightarrow the solutions are linearly independent for all values of t .

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