Math 211

Lecture #25

The Exponential of a Matrix

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Homogeneous Systems

- These are systems of the form

$$\mathbf{x}' = A\mathbf{x},$$

  where A is an $n \times n$ matrix.

- We are looking primarily at homogeneous systems with constant coefficients.

Structure of the Solution Space

Theorem: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_n(t)$ are linearly independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ on the interval I. Then every solution is a linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_n(t)$.

- That is, if $\mathbf{x}(t)$ is a solution, then there are constants C_1, C_2, . . . , and C_n such that

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) + \cdots + C_n\mathbf{x}_n(t).$$

Solution Strategy

- The obvious strategy for completely solving the system is to look for n linearly independent solutions. solutions.

Definition: A set of n linear independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ is called a *fundamental set of solutions.*

- We will look for fundamental sets of solutions.

Exponential of a Matrix

Definition: The *exponential* of the $n \times n$ matrix A is the $n \times n$ matrix

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$
$$= \sum_0^\infty \frac{1}{n!}A^n.$$

- Example: $e^{\lambda I} = e^\lambda I$.

Theorem: The solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}$$

is given by $\mathbf{x}(t) = e^{tA}\mathbf{v}$.

Truncation

- If $\mathbf{v}$ is a vector, we have

$$e^{tA}\mathbf{v} = \mathbf{v} + tA\mathbf{v} + \frac{t^2}{2!}A^2\mathbf{v} + \frac{t^3}{3!}A^3\mathbf{v} + \cdots$$

  - If $A\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = \mathbf{v}$.
  - If $A^2\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = \mathbf{v} + tA\mathbf{v}$.
  - If $A^k\mathbf{v} = \mathbf{0}$, then

$$e^{tA}\mathbf{v} = \mathbf{v} + tA\mathbf{v} + \frac{t^2}{2!}A^2\mathbf{v} + \cdots + \frac{t^{k-1}}{(k-1)!}A^{k-1}\mathbf{v}.$$

- We can compute $e^{tA}\mathbf{v}$ if $\mathbf{v}$ is in the nullspace of A or of a power of A.

Example 1

Consider

$$A = \begin{pmatrix} -6 & 4 \\ -12 & 8 \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

- $A\mathbf{v} = \mathbf{0}$. The series for $e^{tA}\mathbf{v}$ truncates

$$\begin{aligned} e^{tA}\mathbf{v} &= I\mathbf{v} + tA\mathbf{v} + \cdots \\ &= \mathbf{v}. \end{aligned}$$

- Although we cannot compute e^{tA}, we can compute $e^{tA}\mathbf{v}$.

The Law of Exponents

Proposition: Suppose that A and B are $n \times n$ matrices.

1. $e^{A+B} = e^A e^B \quad \Leftrightarrow \quad AB = BA$ (A and B commute).
2. The exponential e^A is invertible and its inverse is e^{-A}.

Suppose that A an $n \times n$ matrix, and λ a number

- Then $A = \lambda I + [A - \lambda I]$, and λI & $A - \lambda I$ commute.
- Therefore $e^{tA} = e^{\lambda t I} e^{t[A-\lambda I]} = e^{\lambda t} e^{t[A-\lambda I]}$.
  - ♦ This simple identity will enable us to compute the exponentials we need. We will link this identity with previous examples.

Propostion

Proposition: Suppose that A is an $n \times n$ matrix, λ is a number, and $\mathbf{v}$ is a vector.

1. If $[A - \lambda I]\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$.
2. If $[A - \lambda I]^2\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}(\mathbf{v} + t[A - \lambda I]\mathbf{v})$.
3. If $[A - \lambda I]^k\mathbf{v} = \mathbf{0}$, then

$$e^{tA}\mathbf{v} = e^{\lambda t}\left(\mathbf{v} + t[A - \lambda I]\mathbf{v} + \frac{t^2}{2!}[A - \lambda I]^2\mathbf{v} + \cdots \right.$$
$$\left. + \frac{t^{k-1}}{(k-1)!}[A - \lambda I]^{k-1}\mathbf{v}\right).$$

Example 2

Consider

$$A = \begin{pmatrix} 7 & 4 \\ -8 & -5 \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

- $A\mathbf{v}_1 = -\mathbf{v}_1$ so

$$\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^{-t}\mathbf{v}_1 = e^{-t}\begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

- $A\mathbf{v}_2 = 3\mathbf{v}_2$ so

$$\mathbf{x}_2(t) = e^{tA}\mathbf{v}_2 = e^{3t}\mathbf{v}_2 = e^{3t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

- Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental set of solutions to $\mathbf{x}' = A\mathbf{x}$.

The Impact of the Proposition

- We can compute $e^{tA}\mathbf{v}$ whenever $\mathbf{v}$ is in the nullspace of $A - \lambda I$ or of some power of $A - \lambda I$ for some number λ.
- Part 1 of the Proposition was all we needed in the example.
- Part 1 of the Proposition is all we will need for most matrices.
- We will use the rest of the Proposition to handle exceptional cases.
- Notice that

$$[A - \lambda I]\mathbf{v} = \mathbf{0} \quad \Leftrightarrow \quad A\mathbf{v} = \lambda\mathbf{v}.$$

Return

Eigenvalues & Eigenvectors

Definition: λ is an *eigenvalue* of A if there is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$. If λ is an eigenvalue of A, then any vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ is called an *eigenvector associated with λ*.

- If λ an eigenvalue of A, and $\mathbf{v}$ is an associated nonzero eigenvector, then $\mathbf{x}(t) = e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$ is a solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}.$$

  - Solutions of the form $e^{\lambda t}\mathbf{v}$ are called *exponential solutions*.

- How do we find eigenvalues and eigenvectors?

Finding Eigenvalues

λ is an eigenvalue of A

$\Leftrightarrow$ there is a vector $\mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

$\Leftrightarrow$ there is a vector $\mathbf{v} \neq \mathbf{0}$ such that $[A - \lambda I]\mathbf{v} = \mathbf{0}$

$\Leftrightarrow$ $A - \lambda I$ has a nontrivial nullspace.

$\Leftrightarrow$ $A - \lambda I$ is singular.

$\Leftrightarrow$ $\det(A - \lambda I) = 0$.

Example part 1

$$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -4-\lambda & 2 \\ -3 & 1-\lambda \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (-4-\lambda)(1-\lambda) + 6 \\ &= \lambda^2 + 3\lambda + 2 \\ &= (\lambda+1)(\lambda+2) \end{aligned}$$

- A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

The Characteristic Polynomial

- If A is an $n \times n$ matrix $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree n.

Definition: The *characteristic polynomial* of the $n \times n$ matrix A is

$$p(\lambda) = \det(A - \lambda I).$$

The *characteristic equation* is $p(\lambda) = 0$.

- Thus, the eigenvalues of A are the roots of the characteristic equation.

Finding Eigenvectors

$\mathbf{v}$ is an eigenvector associated with the eigenvalue λ if

$$A\mathbf{v} = \lambda\mathbf{v}$$
$$\Leftrightarrow (A - \lambda I)\mathbf{v} = \mathbf{0}$$
$$\Leftrightarrow \mathbf{v} \in \text{null}(A - \lambda I)$$

- The set of all eigenvectors associated to the eigenvalue λ is equal to the nullspace of $A - \lambda I$.
  - It is a subspace of $\mathbf{R}^n$.
  - It is called the *eigenspace* of λ.

Example part 2

The matrix $A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$ has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

- $\lambda_1 = -1$: $A - \lambda_1 I = \begin{pmatrix} -4+1 & 2 \\ -3 & 1+1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix}$
  - $\mathbf{v}_1 = (2,3)^T$ is an eigenvector .
  - $\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^{\lambda_1 t}\mathbf{v}_1 = e^{-t}(2,3)^T$ is a solution .
- $\lambda_2 = -2$: $A - \lambda_2 I = \begin{pmatrix} -4+2 & 2 \\ -3 & 1+2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix}$
  - $\mathbf{v}_2 = (1,1)^T$ is an eigenvector .
  - $\mathbf{x}_2(t) = e^{tA}\mathbf{v}_2 = e^{\lambda_2 t}\mathbf{v}_2 = e^{-2t}(1,1)^T$ is a solution .

Example part 3

- $x_1(0) = \mathbf{v}_1$ and $\mathbf{x}_2(0) = \mathbf{v}_2$ are linearly independent .
- $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental set of solutions.
- The general solution is the set of all linear combinations:

$$\begin{aligned}\mathbf{x}(t) &= C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)\\ &= C_1 e^{-t}\begin{pmatrix}2\\3\end{pmatrix} + C_2 e^{-2t}\begin{pmatrix}1\\1\end{pmatrix}\\ &= \begin{pmatrix}2C_1e^{-t} + C_2e^{-2t}\\ 3C_1e^{-t} + C_2e^{-2t}\end{pmatrix}\end{aligned}$$

Our Solution Strategy for $\mathbf{x}' = A\mathbf{x}$

If A is $n \times n$, we are looking for n linearly independent solutions.

- Each eigenvalue λ of A has by definition an associated nonzero eigenvector $\mathbf{v}$. This gives us the solution, $\mathbf{x}(t) = e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$.
- The eigenvalues of A are the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I) = 0$.
  - $p(\lambda)$ has degree n, and usually has n roots.
- Therefore, there are usually n different solutions.
  - Are they linearly independent?

Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A, which are the roots of $\det(A - \lambda I) = 0$.
- For each eigenvalue λ find the eigenspace, which is equal to $\text{null}(A - \lambda I)$.
- If λ is an eigenvalue and $\mathbf{v}$ is an associated nonzero eigenvector, $\mathbf{x}(t) = e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$ is a solution.
- Show that n of these are linearly independent, *if we can.*
  - This must be explored further.

Solving $\mathbf{x}' = A\mathbf{x}$

Cases to be Considered

- Distinct real eigenvalues.
  - In this case the method works as described.
- Complex eigenvalues.
  - The method yields complex solutions, but we will want real solutions.
- Repeated eigenvalues.
  - The method does not always give enough solutions.
  - This is the hard case, but we have not used all of the Proposition.

Linear Independence

Definition: A set of k solutions to the linear system $\mathbf{x}' = A\mathbf{x}$ is *linearly independent* if they are linearly independent at one value of t.

- Proposition $\Rightarrow$ the solutions are linearly independent for all values of t.