

# Math 211

Lecture #26

Solutions of a Planar System

October 27, 2003

## Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of  $A$ , which are the roots of  $\det(A - \lambda I) = 0$ .
- For each eigenvalue  $\lambda$  find the eigenspace, which is equal to  $\text{null}(A - \lambda I)$ .
- If  $\lambda$  is an eigenvalue and  $\mathbf{v}$  is an associated nonzero eigenvector,  $\mathbf{x}(t) = e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$  is a solution.
- Show that  $n$  of these are linearly independent, *if we can*.
  - ♦ This must be explored further.

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## Proposition

**Proposition:** Suppose that  $A$  is an  $n \times n$  matrix,  $\lambda$  is a number, and  $\mathbf{v}$  is a vector.

1. If  $[A - \lambda I]\mathbf{v} = \mathbf{0}$ , then  $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$ .
2. If  $[A - \lambda I]^2\mathbf{v} = \mathbf{0}$ , then  $e^{tA}\mathbf{v} = e^{\lambda t}(\mathbf{v} + t[A - \lambda I]\mathbf{v})$ .
3. If  $[A - \lambda I]^k\mathbf{v} = \mathbf{0}$ , then

$$e^{tA}\mathbf{v} = e^{\lambda t} \left( \mathbf{v} + t[A - \lambda I]\mathbf{v} + \frac{t^2}{2!}[A - \lambda I]^2\mathbf{v} + \cdots + \frac{t^{k-1}}{(k-1)!}[A - \lambda I]^{k-1}\mathbf{v} \right).$$

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### Planar System $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

- The Characteristic Polynomial

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \lambda^2 - T\lambda + D, \end{aligned}$$

where

- $D = a_{11}a_{22} - a_{12}a_{21} = \det(A)$
- $T = a_{11} + a_{22} = \text{tr}(A)$  is the *trace* of  $A$ .
  - ♦ The *trace* of a matrix is the sum of its diagonal elements.

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### The Eigenvalues of $A$

- The eigenvalues of  $A$  are the roots of the characteristic equation  $p(\lambda) = \lambda^2 - T\lambda + D = 0$ .

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- There are three cases:
  - ♦ 2 distinct real roots if  $T^2 - 4D > 0$
  - ♦ 2 complex conjugate roots if  $T^2 - 4D < 0$
  - ♦ Double real root if  $T^2 - 4D = 0$

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### Eigenvectors are Linearly Independent

The problem of determining that solutions are linearly independent is eased by the following result.

**Proposition:** Suppose that  $\lambda_1 \neq \lambda_2$  are eigenvalues of the  $n \times n$  matrix  $A$ , and that  $\mathbf{v}_1 \neq 0$  and  $\mathbf{v}_2 \neq 0$  are eigenvectors associated with  $\lambda_1$  and  $\lambda_2$ , respectively. Then  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are linearly independent.

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## Two Distinct Real Eigenvalues

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2}, \quad \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

- $T^2 - 4D > 0$  so  $\lambda_1 < \lambda_2$ .
- There are associated nonzero eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$ .
- Solutions  $\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^{\lambda_1 t}\mathbf{v}_1$  and  $\mathbf{x}_2(t) = e^{tA}\mathbf{v}_2 = e^{\lambda_2 t}\mathbf{v}_2$ .
- $\mathbf{x}_1(0) = \mathbf{v}_1$  and  $\mathbf{x}_2(0) = \mathbf{v}_2$  are linearly independent;  $\mathbf{x}_1(t)$  and  $\mathbf{x}_2(t)$  form a fundamental set of solutions.
- The general solution is  $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$ .

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Cases

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## Example 1

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -6 & -8 \\ 4 & 6 \end{pmatrix}$$

- Characteristic polynomial:  $p(\lambda) = \lambda^2 - 4$ .
- Eigenvalues:  $\lambda_1 = -2$  and  $\lambda_2 = 2$ .
  - ♦  $\lambda_1 = -2$ . Eigenvector:  $\mathbf{v}_1 = (-2, 1)^T$ .
    - ▶ Solution:  $\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^{\lambda_1 t}\mathbf{v}_1 = e^{-2t}(-2, 1)^T$ .
  - ♦  $\lambda_2 = 2$ . Eigenvector:  $\mathbf{v}_2 = (-1, 1)^T$ .
    - ▶ Solution:  $\mathbf{x}_2(t) = e^{tA}\mathbf{v}_2 = e^{\lambda_2 t}\mathbf{v}_2 = e^{2t}(-1, 1)^T$ .

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## Example 1 Fundamental System

- $\mathbf{x}_1$  and  $\mathbf{x}_2$  are a fundamental set of solutions.
- The general solution is

$$\begin{aligned} \mathbf{x}(t) &= C_1 \mathbf{x}_1(t) + C_2 \mathbf{x}_2(t) \\ &= C_1 e^{-2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}. \end{aligned}$$

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Example

### Example 1 Initial Value Problem

Solve  $\mathbf{x}' = A\mathbf{x}$  with the initial condition  $\mathbf{x}(0) = (1, 4)^T$ .

- We need

$$\mathbf{x}(0) = C_1\mathbf{x}_1(0) + C_2\mathbf{x}_2(0)$$

- $C_1 = -5$  and  $C_2 = 9$ .
- The solution is

$$\begin{aligned}\mathbf{x}(t) &= -5\mathbf{x}_1(t) + 9\mathbf{x}_2(t) \\ &= \begin{pmatrix} 10e^{-2t} - 9e^{2t} \\ -5e^{-2t} + 9e^{2t} \end{pmatrix}.\end{aligned}$$

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IVP

### Complex Eigenvalues

- If the discriminant  $T^2 - 4D < 0$  we have complex eigenvalues

$$\lambda = \frac{T + i\sqrt{4D - T^2}}{2}, \quad \bar{\lambda} = \frac{T - i\sqrt{4D - T^2}}{2}$$

- Example 2:  $\begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$ .

- Characteristic polynomial:  $p(\lambda) = \lambda^2 - 2\lambda + 5$ .
- Eigenvalues:  $\lambda = 1 + 2i$  and  $\bar{\lambda} = 1 - 2i$
- Eigenvectors: ?
  - ▶ We need to know more about complex numbers and matrices.

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Procedure

### Complex Numbers

- $z = x + iy$ ,  $x, y \in \mathbf{R}$ ,  $i^2 = -1$ .
  - $x = \operatorname{Re}(z)$ ,  $y = \operatorname{Im}(z)$ .
  - $\bar{z} = x - iy$  is the *complex conjugate* of  $z$ .
  - $|z| = \sqrt{x^2 + y^2}$  is the *absolute value* of  $z$ .
- Formulas:
  - $x = \operatorname{Re}(z) = \frac{z + \bar{z}}{2}$ ,  $y = \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$ .
  - $\overline{z + w} = \bar{z} + \bar{w}$ ,  $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$ .
  - $|z|^2 = z\bar{z}$ ,  $|zw| = |z||w|$ .
  - $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ ,  $\frac{z}{w} = \frac{z\bar{w}}{|w|^2}$ .

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## The Complex Exponential

- Define:  $e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \dots$ .
  - ♦ Euler's formula:  $e^{iy} = \cos y + i \sin y$ .
  - ♦ Law of exponents:  $e^{z+w} = e^z \cdot e^w$ .
  - ♦  $e^z = e^{x+iy} = e^x e^{iy} = e^x [\cos y + i \sin y]$ .
  - ♦  $|e^z| = e^{\operatorname{Re}(z)}$ ,  $\overline{e^z} = e^{\overline{z}}$ .
- $\frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t}$  even if  $\lambda$  is complex.
- Complex polar coordinates:  $z = x + iy = |z|e^{i\theta}$ , where  $\tan \theta = y/x$ .

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## Complex Matrices

Matrices (or vectors) with complex entries inherit many of the properties of complex numbers.

- $M = A + iB$  where  $A = \operatorname{Re}M$  and  $B = \operatorname{Im}M$  are real matrices.
- $\overline{\overline{M}} = M$ ;  $M = \overline{M} \Leftrightarrow M$  is real.
- $\operatorname{Re}M = \frac{1}{2}(M + \overline{M})$ ;  $\operatorname{Im}M = \frac{1}{2i}(M - \overline{M})$
- $\overline{M + N} = \overline{M} + \overline{N}$
- $\overline{Mz} = \overline{M}\overline{z}$

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## Complex Eigenpairs

$A$  a real matrix

- Suppose that  $\lambda$  is a complex eigenvalue with associated eigenvector  $\mathbf{w}$ . Then  $A\mathbf{w} = \lambda\mathbf{w}$ .
- Conjugating we get

$$\begin{aligned}\overline{A\mathbf{w}} &= \overline{\lambda\mathbf{w}} = A\overline{\mathbf{w}} \\ \overline{\lambda\mathbf{w}} &= \overline{\lambda}\overline{\mathbf{w}}\end{aligned}$$

- $A\mathbf{w} = \lambda\mathbf{w} \Rightarrow \overline{A\mathbf{w}} = \overline{\lambda\mathbf{w}} \Rightarrow A\overline{\mathbf{w}} = \overline{\lambda}\overline{\mathbf{w}}$
- $\Rightarrow \overline{\lambda}$  is an eigenvalue of  $A$  with associated eigenvector  $\overline{\mathbf{w}}$

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Complex Matrices

### Complex Solutions

- Thus complex eigenvalues come in conjugate pairs  $\lambda$  and  $\bar{\lambda}$ .
- The associated eigenvectors also come in conjugate pairs  $\mathbf{w}$  and  $\bar{\mathbf{w}}$ .
- $\lambda \neq \bar{\lambda} \Rightarrow \mathbf{w}$  and  $\bar{\mathbf{w}}$  are linearly independent.
- We get complex exponential solutions

$$\mathbf{z}(t) = e^{\lambda t} \mathbf{w} \quad \text{and} \quad \bar{\mathbf{z}}(t) = e^{\bar{\lambda} t} \bar{\mathbf{w}}.$$

- $\mathbf{z}$  and  $\bar{\mathbf{z}}$  are linearly independent complex valued solutions to  $\mathbf{x}' = A\mathbf{x}$ .

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Complex Matrices

### Real Solutions

$$\mathbf{z}(t) = \mathbf{x}(t) + i\mathbf{y}(t) \quad \& \quad \bar{\mathbf{z}}(t) = \mathbf{x}(t) - i\mathbf{y}(t)$$

$$\mathbf{x}(t) = \operatorname{Re}(\mathbf{z}(t)) = \frac{\mathbf{z}(t) + \bar{\mathbf{z}}(t)}{2}$$

$$\mathbf{y}(t) = \operatorname{Im}(\mathbf{z}(t)) = \frac{\mathbf{z}(t) - \bar{\mathbf{z}}(t)}{2i}$$

- $\mathbf{x}(t)$  and  $\mathbf{y}(t)$  are real valued solutions.
- $\mathbf{x}(t)$  and  $\mathbf{y}(t)$  are linearly independent.

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Complex Matrices

Complex eigenpairs

Complex solutions

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### Complex Solutions in Example 2

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$$

- The eigenvalues are:  $\lambda = 1 + 2i$  and  $\bar{\lambda} = 1 - 2i$ .
- An eigenvector associated to  $\lambda$  is  $\mathbf{w} = (3 - i, 1)^T$ .
- Complex solutions:

$$\mathbf{z}(t) = e^{\lambda t} \mathbf{w} = e^{(1+2i)t} \begin{pmatrix} 3 - i \\ 1 \end{pmatrix}$$

$$\bar{\mathbf{z}}(t) = e^{\bar{\lambda} t} \bar{\mathbf{w}} = e^{(1-2i)t} \begin{pmatrix} 3 + i \\ 1 \end{pmatrix}$$

- This is a fundamental set of solutions.

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### Real Solutions in Example 2

- Real solutions:

$$\mathbf{x}(t) = \operatorname{Re}(\mathbf{z}(t)) = e^t \begin{pmatrix} 3 \cos 2t + \sin 2t \\ \cos 2t \end{pmatrix}$$

$$\mathbf{y}(t) = \operatorname{Im}(\mathbf{z}(t)) = e^t \begin{pmatrix} 3 \sin 2t - \cos 2t \\ \sin 2t \end{pmatrix}$$

- ♦ This is a fundamental set of solutions.

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[Example](#)

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### IVP for Example 2

Solve  $\mathbf{x}' = A\mathbf{x}$ , where

$$A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}, \quad \text{and} \quad \mathbf{x}(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

The solution is

$$\begin{aligned} \mathbf{u}(t) &= 3e^t \begin{pmatrix} 3 \cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} \\ &\quad + 4e^t \begin{pmatrix} 3 \sin 2t - \cos 2t \\ \sin 2t \end{pmatrix} \\ &= e^t \begin{pmatrix} 5 \cos 2t + 15 \sin 2t \\ 3 \cos 2t + 4 \sin 2t \end{pmatrix} \end{aligned}$$

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[Real solutions](#)

### Summary — Complex Eigenvalues

Suppose  $A$  is a real  $2 \times 2$  matrix with

- complex conjugate eigenvalues  $\lambda$  and  $\bar{\lambda}$ , and
- associated nonzero eigenvectors  $\mathbf{w}$  and  $\bar{\mathbf{w}}$ .

Then

- $\mathbf{z}(t) = e^{\lambda t} \mathbf{w}$  and  $\bar{\mathbf{z}}(t) = e^{\bar{\lambda} t} \bar{\mathbf{w}}$  form a complex valued fundamental set of solutions, and
- $\mathbf{x}(t) = \operatorname{Re}(\mathbf{z}(t))$  and  $\mathbf{y}(t) = \operatorname{Im}(\mathbf{z}(t))$  form a real valued fundamental set of solutions.

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### Examples

$$\mathbf{x}' = A\mathbf{x}$$

where

- $A = \begin{pmatrix} 7 & 30 \\ -3 & -11 \end{pmatrix}$
- $A = \begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}$

Summary