Math 211

Lecture #26

Solutions of a Planar System

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Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A, which are the roots of $\det(A - \lambda I) = 0$.
- For each eigenvalue λ find the eigenspace, which is equal to $\text{null}(A - \lambda I)$.
- If λ is an eigenvalue and $\mathbf{v}$ is an associated nonzero eigenvector, $\mathbf{x}(t) = e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$ is a solution.
- Show that n of these are linearly independent, *if we can.*
  - This must be explored further.

Propostion

Proposition: Suppose that A is an $n \times n$ matrix, λ is a number, and $\mathbf{v}$ is a vector.

1. If $[A - \lambda I]\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$.
2. If $[A - \lambda I]^2\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}(\mathbf{v} + t[A - \lambda I]\mathbf{v})$.
3. If $[A - \lambda I]^k\mathbf{v} = \mathbf{0}$, then

$$e^{tA}\mathbf{v} = e^{\lambda t}\left(\mathbf{v} + t[A - \lambda I]\mathbf{v} + \frac{t^2}{2!}[A - \lambda I]^2\mathbf{v} + \cdots \right.$$
$$\left. + \frac{t^{k-1}}{(k-1)!}[A - \lambda I]^{k-1}\mathbf{v}\right).$$

Planar System $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

- The Characteristic Polynomial

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \lambda^2 - T\lambda + D, \end{aligned}$$

where

- $D = a_{11}a_{22} - a_{12}a_{21} = \det(A)$
- $T = a_{11} + a_{22} = \mathrm{tr}(A)$ is the *trace* of A.
  - The *trace* of a matrix is the sum of its diagonal elements.

The Eigenvalues of A

- The eigenvalues of A are the roots of the characteristic equation $p(\lambda) = \lambda^2 - T\lambda + D = 0$.

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- There are three cases:
  - 2 distinct real roots if $T^2 - 4D > 0$
  - 2 complex conjugate roots if $T^2 - 4D < 0$
  - Double real root if $T^2 - 4D = 0$

Eigenvectors are Linearly Independent

The problem of determining that solutions are linearly independent is eased by the following result.

Proposition: Suppose that $\lambda_1 \neq \lambda_2$ are eigenvalues of the $n \times n$ matrix A, and that $\mathbf{v}_1 \neq 0$ and $\mathbf{v}_2 \neq 0$ are eigenvectors associated with λ_1 and λ_2, respectively. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Two Distinct Real Eigenvalues

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2}, \quad \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

- $T^2 - 4D > 0$ so $\lambda_1 < \lambda_2$.
- There are associated nonzero eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.
- Solutions $\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{tA}\mathbf{v}_2 = e^{\lambda_2 t}\mathbf{v}_2$.
- $\mathbf{x}_1(0) = \mathbf{v}_1$ and $\mathbf{x}_2(0) = \mathbf{v}_2$ are linearly independent ; $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ form a fundamental set of solutions.
- The general solution is $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$.

Example 1

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -6 & -8 \\ 4 & 6 \end{pmatrix}$$

- Characteristic polynomial: $p(\lambda) = \lambda^2 - 4$.
- Eigenvalues: $\lambda_1 = -2$ and $\lambda_2 = 2$.
  - $\lambda_1 = -2$. Eigenvector: $\mathbf{v}_1 = (-2, 1)^T$.
    - Solution: $\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^{\lambda_1 t}\mathbf{v}_1 = e^{-2t}(-2, 1)^T$.
  - $\lambda_2 = 2$. Eigenvector: $\mathbf{v}_2 = (-1, 1)^T$.
    - Solution: $\mathbf{x}_2(t) = e^{tA}\mathbf{v}_2 = e^{\lambda_2 t}\mathbf{v}_2 = e^{2t}(-1, 1)^T$.

Example 1 Fundamental System

- $\mathbf{x}_1$ and $\mathbf{x}_2$ are a fundamental set of solutions.
- The general solution is

$$\begin{aligned}\mathbf{x}(t) &= C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)\\ &= C_1 e^{-2t}\begin{pmatrix}-2\\1\end{pmatrix} + C_2 e^{2t}\begin{pmatrix}-1\\1\end{pmatrix}.\end{aligned}$$

Example 1 Initial Value Problem

Solve $\mathbf{x}' = A\mathbf{x}$ with the initial condition $\mathbf{x}(0) = (1, 4)^T$.

- We need

$$\mathbf{x}(0) = C_1\mathbf{x}_1(0) + C_2\mathbf{x}_2(0)$$

  - $C_1 = -5$ and $C_2 = 9$.

- The solution is

$$\begin{aligned} \mathbf{x}(t) &= -5\mathbf{x}_1(t) + 9\mathbf{x}_2(t) \\ &= \begin{pmatrix} 10e^{-2t} - 9e^{2t} \\ -5e^{-2t} + 9e^{2t} \end{pmatrix}. \end{aligned}$$

Complex Eigenvalues

- If the discriminant $T^2 - 4D < 0$ we have complex eigenvalues

$$\lambda = \frac{T + i\sqrt{4D - T^2}}{2}, \quad \bar{\lambda} = \frac{T - i\sqrt{4D - T^2}}{2}$$

- Example 2: $\begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$.
  - Characteristic polynomial: $p(\lambda) = \lambda^2 - 2\lambda + 5$.
  - Eigenvalues: $\lambda = 1 + 2i$ and $\bar{\lambda} = 1 - 2i$
  - Eigenvectors: ?
    - We need to know more about complex numbers and matrices.

Complex Numbers

- $z = x + iy$, $x, y \in \mathbf{R}$, $i^2 = -1$.
  - $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$.
  - $\overline{z} = x - iy$ is the *complex conjugate* of z.
  - $|z| = \sqrt{x^2 + y^2}$ is the *absolute value* of z.
- Formulas:
  - $x = \mathrm{Re}(z) = \dfrac{z + \overline{z}}{2}, \quad y = \mathrm{Im}(z) = \dfrac{z - \overline{z}}{2i}.$
  - $\overline{z + w} = \overline{z} + \overline{w}, \quad \overline{z \cdot w} = \overline{z} \cdot \overline{w}.$
  - $|z|^2 = z\overline{z}, \quad |zw| = |z||w|.$
  - $\dfrac{1}{z} = \dfrac{\overline{z}}{|z|^2}, \quad \dfrac{z}{w} = \dfrac{z\overline{w}}{|w|^2}.$

The Complex Exponential

- Define: $e^z = 1 + z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$.
  - *Euler's formula:* $e^{iy} = \cos y + i \sin y$.
  - Law of exponents: $e^{z+w} = e^z \cdot e^w$.
  - $e^z = e^{x+iy} = e^x e^{iy} = e^x [\cos y + i \sin y]$.
  - $|e^z| = e^{\mathrm{Re}(z)}, \quad \overline{e^z} = e^{\overline{z}}$.
- $\dfrac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t}$ even if λ is complex.
- Complex polar coordinates: $z = x + iy = |z|e^{i\theta}$, where $\tan \theta = y/x$.

Complex Matrices

Matrices (or vectors) with complex entries inherit many of the properties of complex numbers.

- $M = A + iB$ where $A = \mathrm{Re}M$ and $B = \mathrm{Im}M$ are real matrices.
- $\overline{M} = A - iB; \quad M = \overline{M} \Leftrightarrow M$ is real.
- $\mathrm{Re}M = \frac{1}{2}(M + \overline{M}); \quad \mathrm{Im}M = \frac{1}{2i}(M - \overline{M})$
- $\overline{M + N} = \overline{M} + \overline{N}$
- $\overline{M\mathbf{z}} = \overline{M}\overline{\mathbf{z}}$

Complex Eigenpairs

A a real matrix

- Suppose that λ is a complex eigenvalue with associated eigenvector $\mathbf{w}$. Then $A\mathbf{w} = \lambda\mathbf{w}$.
- Conjugating we get

$$\overline{A\mathbf{w}} = \overline{A}\,\overline{\mathbf{w}} = A\overline{\mathbf{w}}$$

$$\overline{\lambda\mathbf{w}} = \overline{\lambda}\,\overline{\mathbf{w}}$$

- $A\mathbf{w} = \lambda\mathbf{w} \Rightarrow \overline{A\mathbf{w}} = \overline{\lambda\mathbf{w}} \Rightarrow A\overline{\mathbf{w}} = \overline{\lambda}\,\overline{\mathbf{w}}$
- $\Rightarrow \overline{\lambda}$ is an eigenvalue of A with associated eigenvector $\overline{\mathbf{w}}$

Complex Solutions

- Thus complex eigenvalues come in conjugate pairs λ and $\overline{\lambda}$.
- The associated eigenvectors also come in conjugate pairs $\mathbf{w}$ and $\overline{\mathbf{w}}$.
- $\lambda \neq \overline{\lambda} \quad \Rightarrow \quad \mathbf{w}$ and $\overline{\mathbf{w}}$ are linearly independent.
- We get complex exponential solutions

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{w} \quad \text{and} \quad \overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}}.$$

- $\mathbf{z}$ and $\overline{\mathbf{z}}$ are linearly independent complex valued solutions to $\mathbf{x}' = A\mathbf{x}$.

Real Solutions

$$\mathbf{z}(t) = \mathbf{x}(t) + i\mathbf{y}(t) \quad \& \quad \overline{\mathbf{z}}(t) = \mathbf{x}(t) - i\mathbf{y}(t)$$

$$\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t)) = \frac{\mathbf{z}(t) + \overline{\mathbf{z}}(t)}{2}$$

$$\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t)) = \frac{\mathbf{z}(t) - \overline{\mathbf{z}}(t)}{2i}$$

- $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are real valued solutions.
- $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are linearly independent.

Complex Solutions in Example 2

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$$

- The eigenvalues are: $\lambda = 1 + 2i$ and $\overline{\lambda} = 1 - 2i$.
- An eigenvector associated to λ is $\mathbf{w} = (3 - i, 1)^T$.
- Complex solutions:

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{w} = e^{(1+2i)t}\begin{pmatrix} 3 - i \\ 1 \end{pmatrix}$$

$$\overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}} = e^{(1-2i)t}\begin{pmatrix} 3 + i \\ 1 \end{pmatrix}$$

♦ This is a fundamental set of solutions.

Real Solutions in Example 2

- Real solutions:

$$\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t)) = e^t \begin{pmatrix} 3\cos 2t + \sin 2t \\ \cos 2t \end{pmatrix}$$

$$\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t)) = e^t \begin{pmatrix} 3\sin 2t - \cos 2t \\ \sin 2t \end{pmatrix}$$

♦ This is a fundamental set of solutions.

IVP for Example 2

Solve $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}, \quad \text{and} \quad \mathbf{x}(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

The solution is

$$\begin{aligned} \mathbf{u}(t) = 3e^t & \begin{pmatrix} 3\cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} \\ & + 4e^t \begin{pmatrix} 3\sin 2t - \cos 2t \\ \sin 2t \end{pmatrix} \\ = e^t & \begin{pmatrix} 5\cos 2t + 15\sin 2t \\ 3\cos 2t + 4\sin 2t \end{pmatrix} \end{aligned}$$

Summary — Complex Eigenvalues

Suppose A is a real 2×2 matrix with

- complex conjugate eigenvalues λ and $\overline{\lambda}$, and
- associated nonzero eigenvectors $\mathbf{w}$ and $\overline{\mathbf{w}}$.

Then

- $\mathbf{z}(t) = e^{\lambda t}\mathbf{w}$ and $\overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}}$ form a complex valued fundamental set of solutions, and
- $\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t))$ and $\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t))$ form a real valued fundamental set of solutions.

Examples

$$\mathbf{x}' = A\mathbf{x}$$

where

- $A = \begin{pmatrix} 7 & 30 \\ -3 & -11 \end{pmatrix}$
- $A = \begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix}$