

Math 211

Lecture #28

Phase Plane Portraits

October 31, 2003

Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A , which are the roots of $\det(A - \lambda I) = 0$.
- For each eigenvalue λ find the eigenspace, which is equal to $\text{null}(A - \lambda I)$.
- If λ is an eigenvalue and \mathbf{v} is an associated nonzero eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Show that n of these are linearly independent, *if we can*.
 - This must be explored further if the system has dimension $n \geq 3$.

Return

Planar System $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

- The characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D.$$

where $T = \text{tr } A$ and $D = \det A$.

- The eigenvalues of A are the roots of $p(\lambda) = \lambda^2 - T\lambda + D$,

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

Return

Procedure

Qualitative Analysis for a Planar System $\mathbf{x}' = A\mathbf{x}$

- Equilibrium points for the system
 - ♦ The set of equilibrium points equals $\text{null}(A)$.
 - ♦ If A is nonsingular, the only equilibrium point is $\mathbf{0}$.
- Can we list the types of all possible equilibrium points for planar linear systems?
 - ♦ The cases are distinguished by different types of solution curves in the phase plane.
 - ♦ We will do the six most important cases.

Return

Cases

Distinct Real Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D > 0$.

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} < \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

- Eigenvectors \mathbf{v}_1 and \mathbf{v}_2 . The general solution is

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2$$

- There are three subcases depending on the signs of the eigenvalues.

Return

Solution

Exponential Solutions

$$\mathbf{x}(t) = C e^{\lambda t} \mathbf{v}$$

- The solution curve is a straight half-line through $C\mathbf{v}$.
Sometimes called *half-line* solutions.
- If $\lambda > 0$ the solution starts at $\mathbf{0}$ for $t = -\infty$, and tends to ∞ as $t \rightarrow \infty$. This is an *unstable solution*
- If $\lambda < 0$ the solution starts at ∞ for $t = -\infty$, and tends to $\mathbf{0}$ as $t \rightarrow \infty$. This is a *stable solution*

Return

Real case

Saddle Point

- $\lambda_1 < 0 < \lambda_2$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2$
- Two stable exponential solutions ($C_2 = 0$)
- Two unstable exponential solutions ($C_1 = 0$).
- Suppose that $C_1 \neq 0$ and $C_2 \neq 0$.
 - ♦ As $t \rightarrow \infty$, $\mathbf{x}(t) \rightarrow \infty$, approaching the half-line through $C_2 \mathbf{v}_2$.
 - ♦ As $t \rightarrow -\infty$, $\mathbf{x}(t) \rightarrow \infty$, approaching the half-line through $C_2 \mathbf{v}_1$.

Return

Real eigenvalues

Nodal Sink

- $\lambda_1 < \lambda_2 < 0$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2$
- Four stable exponential solutions.
- All solutions $\rightarrow \mathbf{0}$ as $t \rightarrow \infty$. (Stable)
 - ♦ Tangent to $C_2 \mathbf{v}_2$ if $C_2 \neq 0$.
- All solutions $\rightarrow \infty$ as $t \rightarrow -\infty$.
 - ♦ \parallel to the half line through $C_1 \mathbf{v}_1$ if $C_1 \neq 0$.

Return

Real eigenvalues

Nodal Source

- $0 < \lambda_1 < \lambda_2$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2$
- Four unstable exponential solutions.
- All solutions $\rightarrow \mathbf{0}$ as $t \rightarrow -\infty$.
 - ♦ Tangent to $C_1 \mathbf{v}_1$ if $C_1 \neq 0$.
- All solutions $\rightarrow \infty$ as $t \rightarrow \infty$. (Unstable)
 - ♦ \parallel to the half line through $C_2 \mathbf{v}_2$ if $C_2 \neq 0$.

Return

Real eigenvalues

Nodal Sink

Complex Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D < 0$

$$\lambda = \alpha + i\beta \quad \text{and} \quad \bar{\lambda} = \alpha - i\beta.$$

- Eigenvector $\mathbf{w} = \mathbf{v}_1 + i\mathbf{v}_2$ associated to λ .
- Complex solutions

$$\mathbf{z}(t) = e^{\lambda t} \mathbf{w} = e^{t(\alpha+i\beta)} [\mathbf{v}_1 + i\mathbf{v}_2]$$

$$\bar{\mathbf{z}}(t) = e^{\bar{\lambda} t} \bar{\mathbf{w}} = e^{t(\alpha-i\beta)} [\mathbf{v}_1 - i\mathbf{v}_2]$$

Return

Solution

- Real solutions

$$\mathbf{x}_1(t) = \text{Re}(\mathbf{z}(t)) = e^{\alpha t} [\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2]$$

$$\mathbf{x}_2(t) = \text{Im}(\mathbf{z}(t)) = e^{\alpha t} [\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2]$$

- General solution

$$\begin{aligned} \mathbf{x}(t) = & C_1 e^{\alpha t} [\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2] \\ & + C_2 e^{\alpha t} [\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2] \end{aligned}$$

- There are three cases depending on the sign of $\alpha = \text{Re}(\lambda)$.

Return

Solution

Center

- $\alpha = \text{Re}(\lambda) = 0$

- The general real solution is

$$\begin{aligned} \mathbf{x}(t) = & C_1 [\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2] \\ & + C_2 [\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2] \end{aligned}$$

- Every solution is periodic with period $T = 2\pi/\beta$.
- All solution curves are ellipses.

Return

Spiral Sink

- $\alpha = \operatorname{Re}(\lambda) < 0$
- The general real solution is

$$\mathbf{x}(t) = C_1 e^{\alpha t} [\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2] \\ + C_2 e^{\alpha t} [\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2]$$

- All solutions spiral into $\mathbf{0}$ as $t \rightarrow \infty$.

Return

Spiral Source

- $\alpha = \operatorname{Re}(\lambda) > 0$
- The general real solution is

$$\mathbf{x}(t) = C_1 e^{\alpha t} [\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2] \\ + C_2 e^{\alpha t} [\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2]$$

- All solutions spiral into $\mathbf{0}$ as $t \rightarrow -\infty$.

Return

Planar Systems

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- The eigenvalues are the roots of the characteristic polynomial $p(\lambda) = \lambda^2 - T\lambda + D$, so

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \\ = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2$$

- Hence, $T = \lambda_1 + \lambda_2$ and $D = \lambda_1\lambda_2$.
- Duality between (λ_1, λ_2) and (T, D) .
- We will represent a system by the location of (T, D) in the TD -plane — the *trace-determinant plane*.

Return

Planar Systems

Trace-Determinant Plane

- $T^2 - 4D > 0$
 - ♦ \Rightarrow distinct real eigenvalues λ_1 & λ_2
 - ♦ $D = \lambda_1 \lambda_2 < 0 \Rightarrow$ Saddle point.
 - ♦ $D = \lambda_1 \lambda_2 > 0 \Rightarrow$ Eigenvalues have the same sign.
 - ▶ $T = \lambda_1 + \lambda_2 > 0 \Rightarrow$ Nodal source.
 - ▶ $T = \lambda_1 + \lambda_2 < 0 \Rightarrow$ Nodal sink.

Return

Duality

- $T^2 - 4D < 0 \Rightarrow$ complex eigenvalues
 - $\lambda = \alpha + i\beta$ and $\bar{\lambda} = \alpha - i\beta$.
 - ♦ $T = \lambda + \bar{\lambda} = 2\alpha > 0 \Rightarrow$ Spiral source.
 - ♦ $T = \lambda + \bar{\lambda} = 2\alpha < 0 \Rightarrow$ Spiral sink.
 - ♦ $T = \lambda + \bar{\lambda} = 2\alpha = 0 \Rightarrow$ Center.

Return

Duality

TD plane

Cases

Types of Equilibrium Points

- *Generic types*
 - ♦ Saddle, nodal source, nodal sink, spiral source, and spiral sink.
 - ♦ All occupy large open subsets of the trace-determinant plane.
- *Nongeneric types*
 - ♦ Center and many others. Occupy pieces of the boundaries between the generic types.

Return

Real Distinct Eigenvalues

Suppose A is a real 2×2 matrix with real eigenvalues $\lambda_1 \neq \lambda_2$, and associated nonzero eigenvectors \mathbf{v}_1 and \mathbf{v}_2 .

Then $\mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda_2 t} \mathbf{v}_2$ form a fundamental set of solutions.

Return

Cases

Complex Eigenvalues

Suppose A is a real 2×2 matrix with complex conjugate eigenvalues λ and $\bar{\lambda}$, and associated nonzero eigenvectors \mathbf{w} and $\bar{\mathbf{w}}$.

Then

- $\mathbf{z}(t) = e^{\lambda t} \mathbf{w}$ and $\bar{\mathbf{z}}(t) = e^{\bar{\lambda} t} \bar{\mathbf{w}}$ form a complex valued fundamental set of solutions, and
- $\mathbf{x}(t) = \operatorname{Re}(\mathbf{z}(t))$ and $\mathbf{y}(t) = \operatorname{Im}(\mathbf{z}(t))$ form a real valued fundamental set of solutions.

Return

Cases

The Degenerate Planar Case

- Find the (only) eigenvalue λ .
- $[A - \lambda I]^2 = 0I$, so $e^{tA} = e^{\lambda t} (I + t[A - \lambda I])$.
 - ♦ The solution to the initial value problem $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x} = \mathbf{x}_0$ is $\mathbf{x}(t) = e^{tA} \mathbf{x}_0$.
- To get a fundamental set of solutions solve the IVP with any two linearly independent initial vectors.
 - ♦ The column vectors in e^{tA} .
 - ♦ $\mathbf{x}_1(t) = e^{\lambda t} \mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda t} [\mathbf{v}_2 + t\mathbf{v}_1]$, where \mathbf{v}_1 is a nonzero eigenvector, and \mathbf{v}_2 is chosen so that $[A - \lambda I]\mathbf{v}_2 = \mathbf{v}_1$.

Return

Cases