Math 211

Lecture #28

Phase Plane Portraits

October 31, 2003

Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A, which are the roots of $\det(A - \lambda I) = 0$.
- For each eigenvalue λ find the eigenspace, which is equal to $\text{null}(A - \lambda I)$.
- If λ is an eigenvalue and $\mathbf{v}$ is an associated nonzero eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Show that n of these are linearly independent, *if we can.*
  - This must be explored further if the system has dimension $n \geq 3$.

Planar System $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

- The characteristic polynomial is

$$p(\lambda) = \lambda^2 - T\lambda + D.$$

where $T = \operatorname{tr} A$ and $D = \det A$.

- The eigenvalues of A are the roots of $p(\lambda) = \lambda^2 - T\lambda + D$,

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

Qualitative Analysis for a Planar System $\mathbf{x}' = A\mathbf{x}$

- Equilibrium points for the system
  - The set of equilibrium points equals $\mathrm{null}(A)$.
  - If A is nonsingular, the only equilibrium point is $\mathbf{0}$.
- Can we list the types of all possible equilibrium points for planar linear systems?
  - The cases are distinguished by different types of solution curves in the phase plane.
  - We will do the six most important cases.

Distinct Real Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D > 0$.

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} < \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

- Eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The general solution is

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

- There are three subcases depending on the signs of the eigenvalues.

Exponential Solutions

$$\mathbf{x}(t) = Ce^{\lambda t}\mathbf{v}$$

- The solution curve is a straight half-line through $C\mathbf{v}$. Sometimes called *half-line* solutions.
- If $\lambda > 0$ the solution starts at $\mathbf{0}$ for $t = -\infty$, and tends to ∞ as $t \to \infty$. This is an *unstable solution*
- If $\lambda < 0$ the solution starts at ∞ for $t = -\infty$, and tends to $\mathbf{0}$ as $t \to \infty$. This is a *stable solution*

Saddle Point

- $\lambda_1 < 0 < \lambda_2$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$
- Two stable exponential solutions $(C_2 = 0)$
- Two unstable exponential solutions $(C_1 = 0)$.
- Suppose that $C_1 \neq 0$ and $C_2 \neq 0$.
  - As $t \to \infty$, $\mathbf{x}(t) \to \infty$, approaching the half-line through $C_2\mathbf{v}_2$.
  - As $t \to -\infty$, $\mathbf{x}(t) \to \infty$, approaching the half-line through $C_2\mathbf{v}_1$.

Nodal Sink

- $\lambda_1 < \lambda_2 < 0$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$
- Four stable exponential solutions.
- All solutions $\to \mathbf{0}$ as $t \to \infty$. (Stable)
  - Tangent to $C_2\mathbf{v}_2$ if $C_2 \neq 0$.
- All solutions $\to \infty$ as $t \to -\infty$.
  - $\|$ to the half line through $C_1\mathbf{v}_1$ if $C_1 \neq 0$.

Nodal Source

- $0 < \lambda_1 < \lambda_2$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$
- Four unstable exponential solutions.
- All solutions $\to \mathbf{0}$ as $t \to -\infty$.
  - Tangent to $C_1\mathbf{v}_1$ if $C_1 \neq 0$.
- All solutions $\to \infty$ as $t \to \infty$. (Unstable)
  - $\|$ to the half line through $C_2\mathbf{v}_2$ if $C_2 \neq 0$.

Complex Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D < 0$

$$\lambda = \alpha + i\beta \quad \text{and} \quad \overline{\lambda} = \alpha - i\beta.$$

- Eigenvector $\mathbf{w} = \mathbf{v}_1 + i\mathbf{v}_2$ associated to λ.
- Complex solutions

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{w} = e^{t(\alpha+i\beta)}[\mathbf{v}_1 + i\mathbf{v}_2]$$
$$\overline{\mathbf{z}}(t) = e^{\overline{\lambda}t}\overline{\mathbf{w}} = e^{t(\alpha-i\beta)}[\mathbf{v}_1 - i\mathbf{v}_2]$$

- Real solutions

$$\mathbf{x}_1(t) = \mathrm{Re}(\mathbf{z}(t)) = e^{\alpha t}[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2]$$

$$\mathbf{x}_2(t) = \mathrm{Im}(\mathbf{z}(t)) = e^{\alpha t}[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]$$

- General solution

$$\begin{aligned}\mathbf{x}(t) = C_1 e^{\alpha t}&[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2] \\ &+ C_2 e^{\alpha t}[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]\end{aligned}$$

- There are three cases depending on the sign of $\alpha = \mathrm{Re}(\lambda)$.

Center

- $\alpha = \mathrm{Re}(\lambda) = 0$
- The general real solution is

$$\mathbf{x}(t) = C_1[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2] + C_2[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]$$

- Every solution is periodic with period $T = 2\pi/\beta$.
- All solution curves are ellipses.

Spiral Sink

- $\alpha = \text{Re}(\lambda) < 0$
- The general real solution is

$$\mathbf{x}(t) = C_1 e^{\alpha t}[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2] + C_2 e^{\alpha t}[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]$$

- All solutions spiral into $\mathbf{0}$ as $t \to \infty$.

Spiral Source

- $\alpha = \mathrm{Re}(\lambda) > 0$
- The general real solution is

$$\mathbf{x}(t) = C_1 e^{\alpha t}[\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2] + C_2 e^{\alpha t}[\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2]$$

- All solutions spiral into $\mathbf{0}$ as $t \to -\infty$.

Planar Systems

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- The eigenvalues are the roots of the characteristic polynomial$p(\lambda) = \lambda^2 - T\lambda + D$, so

$$\begin{aligned} p(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2) \\ &= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \end{aligned}$$

- Hence, $T = \lambda_1 + \lambda_2$ and $D = \lambda_1\lambda_2$.
- Duality between $(\lambda_1,\ \lambda_2)$ and (T, D).
- We will represent a system by the location of (T, D) in the TD-plane — the *trace-determinant plane.*

Trace-Determinant Plane

- $T^2 - 4D > 0$
  - $\Rightarrow$ distinct real eigenvalues $\lambda_1 \,\&\, \lambda_2$
  - $D = \lambda_1\lambda_2 < 0 \Rightarrow$ Saddle point.
  - $D = \lambda_1\lambda_2 > 0 \Rightarrow$ Eigenvalues have the same sign.
    - $T = \lambda_1 + \lambda_2 > 0 \Rightarrow$ Nodal source.
    - $T = \lambda_1 + \lambda_2 < 0 \Rightarrow$ Nodal sink.

- $T^2 - 4D < 0 \Rightarrow$ complex eigenvalues

$$\lambda = \alpha + i\beta \quad \text{and} \quad \overline{\lambda} = \alpha - i\beta.$$

  - $T = \lambda + \overline{\lambda} = 2\alpha > 0 \Rightarrow$ Spiral source.
  - $T = \lambda + \overline{\lambda} = 2\alpha < 0 \Rightarrow$ Spiral sink.
  - $T = \lambda + \overline{\lambda} = 2\alpha = 0 \Rightarrow$ Center.

Types of Equilibrium Points

- *Generic* types
  - Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - All occupy large open subsets of the trace-determinant plane.
- *Nongeneric* types
  - Center and many others. Occupy pieces of the boundaries between the generic types.

Real Distinct Eigenvalues

Suppose A is a real 2×2 matrix with real eigenvalues $\lambda_1 \neq \lambda_2$, and associated nonzero eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Then $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda_1 t}\mathbf{v}_2$ form a fundamental set of solutions.

Complex Eigenvalues

Suppose A is a real 2×2 matrix with complex conjugate eigenvalues λ and $\overline{\lambda}$, and associated nonzero eigenvectors $\mathbf{w}$ and $\overline{\mathbf{w}}$.

Then

- $\mathbf{z}(t) = e^{\lambda t}\mathbf{w}$ and $\overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}}$ form a complex valued fundamental set of solutions, and
- $\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t))$ and $\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t))$ form a real valued fundamental set of solutions.

The Degenerate Planar Case

- Find the (only) eigenvalue λ.
- $[A - \lambda I]^2 = 0I$, so $e^{tA} = e^{\lambda t}\left(I + t[A - \lambda I]\right)$.
  - The solution to the intial value problem $\mathbf{x}' = A\mathbf{x}$ with $\mathbf{x} = \mathbf{x}_0$ is $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$.
- To get a fundamental set of solutions solve the IVP with any two linearly independent initial vectors.
  - The column vectors in e^{tA}.
  - $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda t}[\mathbf{v}_2 + t\mathbf{v}_1]$, where $\mathbf{v}_1$ is a nonzero eigenvector, and $\mathbf{v}_2$ is chosen so that $[A - \lambda I]\mathbf{v}_2 = \mathbf{v}_1$.