Math 211

Lecture #29

Systems of Higher Dimension

November 3, 2003

Higher Dimensional Systems

$$\mathbf{x}' = A\mathbf{x}$$

- A is a real $n \times n$ matrix.
- If λ is an eigenvalue and $\mathbf{v} \neq 0$ is an associated eigenvector, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Much like the planar case, but now we need n linearly independent solutions.
- We no longer have the easy way to compute the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$.

Return

Proposition: Suppose that $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of A, and that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are associated nonzero eigenvectors. Then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

Theorem: Suppose the $n \times n$ real matrix A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, and that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are associated nonzero eigenvectors. Then the exponential solutions $\mathbf{x}_i(t) = e^{\lambda_i t}\mathbf{v}_i$, $1 \leq i \leq n$ form a fundamental set of solutions for the system $\mathbf{x}' = A\mathbf{x}$.

Return

Examples:

1. $A = \begin{pmatrix} -2 & 3 & -4 \\ 0 & 1 & 0 \\ 0 & 4 & -1 \end{pmatrix}$
2. $A = \begin{pmatrix} 17 & -30 & -8 \\ 16 & -29 & -8 \\ -12 & 24 & 7 \end{pmatrix}$
   - Use MATLAB.

Complex Eigenvalues

A a real $n \times n$ matrix with a complex eigenvalue λ and associate eigenvector $\mathbf{w}$.

- $\Rightarrow \overline{\lambda}$ is an eigenvalue and $\overline{\mathbf{w}}$ is an associated nonzero eigenvector.
- Complex valued solutions: $\mathbf{z}(t) = e^{\lambda t}\mathbf{w}$

$$\overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}}.$$

- Real solutions: $\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t))$

$$\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t)).$$

Example 3

$$A = \begin{pmatrix} 21 & 10 & 4 \\ -70 & -31 & -10 \\ 30 & 10 & -1 \end{pmatrix}$$

- The theorem applies if some of the eigenvalues are complex and we replace complex conjugate pairs of solutions by their real and imaginary parts.

Repeated Eigenvalues – Example 4

$$A = \begin{pmatrix} -5 & -10 & 6 \\ 8 & 19 & -12 \\ 12 & 30 & -19 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- $\lambda_1 = -3$
  - Eigenspace is spanned by $\mathbf{v}_1 = (-1/3, 2/3, 1)^T \Rightarrow$ one exponential solution

$$\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^{-3t}(-1/3, 2/3, 1)^T$$

- $\lambda_2 = -1$
  - Eigenspace has basis $\mathbf{v}_2 = (-5/2, 1, 0)^T$ and $\mathbf{v}_3 = (3/2, 0, 1)^T$.
  - Linearly independent solutions

$$\mathbf{x}_2(t) = e^{tA}\mathbf{v}_2 = e^{-t}\begin{pmatrix} -5/2 \\ 1 \\ 0 \end{pmatrix} \quad \&$$

$$\mathbf{x}_3(t) = e^{tA}\mathbf{v}_3 = e^{-t}\begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix}$$

- $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are a fundamental set of solutions.

Repeated Eigenvalues – Example 5

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- $\lambda_1 = -3$
  - Eigenspace is spanned by $\mathbf{v}_1 = (-1/2, 3/2, 1)^T \Rightarrow$ one exponential solution

$$\mathbf{x}_1(t) = e^{tA}\mathbf{v}_1 = e^{-3t}(-1/2, 3/2, 1)^T$$

Example 5 part 2

- $\lambda_2 = -1$
  - Eigenspace is spanned by $\mathbf{v}_2 = (-1/2, 1, 1)^T \Rightarrow$ only one exponential solution

$$\mathbf{x}_2(t) = e^{tA}\mathbf{v}_2 = e^{-t}\begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix}$$

- Need a third solution. Compute the nullspace of $[A - \lambda_2 I]^2$.

Example 5 part 3

- The nullspace of $[A - \lambda_2 I]^2$ has dimension 2.
- Choose any vector in $\text{null}([A - \lambda_2 I]^2)$ which is not a multiple of $\mathbf{v}_2$. For example $\mathbf{v}_3 = (1, 0, 0)^T$. Our third solution is

$$\begin{aligned} \mathbf{x}_3(t) &= e^{tA}\mathbf{v}_3 = e^{-t}\left(\mathbf{v}_3 + t[A + I]\mathbf{v}_3\right) \\ &= e^{-t}\begin{pmatrix} 1 + 2t \\ -4t \\ -4t \end{pmatrix}. \end{aligned}$$

Multiplicities

A an $n \times n$ matrix

- Distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.
- The characteristic polynomial is

$$p(\lambda) = (\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2} \cdot \ldots \cdot (\lambda - \lambda_k)^{q_k}.$$

- The *algebraic multiplicity* of λ_j is q_j.
- The *geometric multiplicity* of λ_j is d_j, the dimension of the eigenspace of λ_j.

Properties of Multiplicities

- $q_1 + q_2 + \cdots + q_k = n$.
- $1 \leq d_j \leq q_j$.
- There are d_j linearly independent exponential solutions corresponding to λ_j.
- If $d_j = q_j$ for all j we have n linearly independent exponential solutions.
- If $d_j < q_j$ we can use our Proposition.

Generalized Eigenvectors

Definition: If λ is an eigenvalue of A and $[A - \lambda I]^p \mathbf{v} = \mathbf{0}$ for some integer $p \geq 1$, then $\mathbf{v}$ is called a *generalized eigenvector* associated with λ.

Theorem: If λ is an eigenvalue of A with algebraic multiplicity q, then there is an integer $p \leq q$ such that $\mathrm{null}([A - \lambda I]^p)$ has dimension q.

- For each generalized eigenvector $\mathbf{v}$ we can compute $e^{tA}\mathbf{v}$.
- We can find q linearly independent solutions associated with the eigenvalue λ.

Procedure for λ of algebraic multiplicity q

To find q linearly independent solutions associated with λ:

- Find the smallest integer p such that $\mathrm{null}([A - \lambda I]^p)$ has dimension q.
- Find a basis $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_q$ of $\mathrm{null}([A - \lambda I]^p)$.
- For $j = 1, 2, \ldots, q$ compute $\mathbf{x}_j(t) = e^{tA}\mathbf{v}_j$.

Propostion

Proposition: Suppose that A is an $n \times n$ matrix, λ is a number, and $\mathbf{v}$ is a vector.

1. If $[A - \lambda I]\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$.
2. If $[A - \lambda I]^2\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}(\mathbf{v} + t[A - \lambda I]\mathbf{v})$.
3. If $[A - \lambda I]^k\mathbf{v} = \mathbf{0}$, then

$$e^{tA}\mathbf{v} = e^{\lambda t}\left(\mathbf{v} + t[A - \lambda I]\mathbf{v} + \frac{t^2}{2!}[A - \lambda I]^2\mathbf{v} + \cdots + \frac{t^{k-1}}{(k-1)!}[A - \lambda I]^{k-1}\mathbf{v}\right).$$

Return