# Math 211

Lecture #30

Solutions of Systems and Stability

November 5, 2003

# Multiplicities

$A$ an $n \times n$ matrix

- Distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.
- The characteristic polynomial is

$$p(\lambda) = (\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2} \cdot \ldots \cdot (\lambda - \lambda_k)^{q_k}.$$

- The *algebraic multiplicity* of $\lambda_j$ is $q_j$.
- The *geometric multiplicity* of $\lambda_j$ is $d_j$, the dimension of the eigenspace of $\lambda_j$.

# Properties of Multiplicities

- $q_1 + q_2 + \cdots + q_k = n$.
- $1 \leq d_j \leq q_j$.
- There are $d_j$ linearly independent exponential solutions corresponding to $\lambda_j$.
- If $d_j = q_j$ for all $j$ we have $n$ linearly independent exponential solutions.
- If $d_j < q_j$ we can use our Proposition.

# Generalized Eigenvectors

**Definition:** If $\lambda$ is an eigenvalue of $A$ and $[A - \lambda I]^p \mathbf{v} = \mathbf{0}$ for some integer $p \geq 1$, then $\mathbf{v}$ is called a *generalized eigenvector* associated with $\lambda$.

**Theorem:** If $\lambda$ is an eigenvalue of $A$ with algebraic multiplicity $q$, then there is an integer $p \leq q$ such that $\mathrm{null}([A - \lambda I]^p)$ has dimension $q$.

- For each generalized eigenvector $\mathbf{v}$ we can compute $e^{tA}\mathbf{v}$.
- We can find $q$ linearly independent solutions associated with the eigenvalue $\lambda$.

# Procedure for Solving $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues and their algebraic multiplicities.
- For each eigenvalue $\lambda$ with algebraic multiplicity $q$ find $q$ linearly independent solutions associated with $\lambda$:
  - Find the smallest integer $p$ such that $\text{null}([A - \lambda I]^p)$ has dimension q.
  - Find a basis $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , $\mathbf{v}_q$ of $\text{null}([A - \lambda I]^p)$.
  - For $j = 1, 2, \ldots, q$ compute $\mathbf{x}_j(t) = e^{tA}\mathbf{v}_j$.
- This results in $n$ linearly independent solutions.

# Procedure for a Complex Eigenvalue

- If $\lambda$ is complex of algebraic multiplicity $q$. Then $\overline{\lambda}$ also has multiplicity $q$.
  - Find the smallest integer $p$ such that $\mathrm{null}([A - \lambda I]^p)$ has dimension q.
  - Find a basis $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_q$ of $\mathrm{null}([A - \lambda I]^p)$.
  - For $j = 1, 2, \ldots, q$ compute $\mathbf{z}_j(t) = e^{tA}\mathbf{w}$.
  - Compute $\mathbf{x}_j(t) = \mathrm{Re}(\mathbf{z}_j(t))$ and $\mathbf{y}_j(t) = \mathrm{Im}(\mathbf{z}_j(t))$.
- This results in $2q$ linearly independent real solutions corresponding to the eigenvalues $\lambda$ and $\overline{\lambda}$.

# Stability

Autonomous system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ with an equilibrium point at $\mathbf{x}_0$. The basic question is: What happens to *all solutions* that start near $\mathbf{x}_0$ as $t \to \infty$?

- $\mathbf{x}_0$ is *stable* if for every $\epsilon > 0$ there is a $\delta > 0$ such that a solution $\mathbf{x}(t)$ with $|\mathbf{x}(0) - \mathbf{x}_0| < \delta \Rightarrow |\mathbf{x}(t) - \mathbf{x}_0| < \epsilon$ for all $t \geq 0$.
  - Every solution that starts close to $\mathbf{x}_0$ stays close to $\mathbf{x}_0$.
  - In dimension 2 centers and sinks are stable.

- $\mathbf{x}_0$ is *asymptotically stable* if it is stable and there is an $\eta > 0$ such that if $\mathbf{x}(t)$ is a solution with $|\mathbf{x}(0) - \mathbf{x}_0| < \eta$, then $\mathbf{x}(t) \to \mathbf{x}_0$ as $t \to \infty$.
  - Every solution that starts close to $\mathbf{x}_0$ approaches $\mathbf{x}_0$.
  - In $d = 2$ sinks are asymptotically stable, centers are not.
  - $\mathbf{x}_0$ is called a *sink*.
- $\mathbf{x}_0$ is *unstable* if there is an $\epsilon > 0$ such that for any $\delta > 0$ there is a solution $\mathbf{x}(t)$ with $|\mathbf{x}(0) - \mathbf{x}_0| < \delta$ with the property that there are values of $t > 0$ such that $|\mathbf{x}(t) - \mathbf{x}_0| > \epsilon$.
  - There are solutions starting arbitrarily close to $\mathbf{x}_0$ that move away from $\mathbf{x}_0$.
  - In $d$ =2 sources and saddles are unstable.

# Dimension 2

- Sinks are asymptotically stable.
  - The eigenvalues have negative real part.
- Sources are unstable.
  - The eigenvalues have positive real part.
- Saddles are unstable.
  - One eigenvalue has positive real part.
- Centers are stable but not asymptotically stable.
  - The eigenvalues have real part $= 0$.

# Stability Theorem

**Theorem:** Let $A$ be an $n \times n$ real matrix.

- Suppose the real part of every eigenvalue of $A$ is negative. Then $\mathbf{0}$ is an asymptotically stable equilibrium point for the system $\mathbf{x}' = A\mathbf{x}$.
- Suppose $A$ has at least one eigenvalue with positive real part. Then $\mathbf{0}$ is an unstable equilibrium point for the system $\mathbf{x}' = A\mathbf{x}$.

Notice that if there are eigenvalues with real part equal to 0, no conclusion is made.

# Examples

- Suppose the dimension is 2 and $T^2 - 4D = 0$.
  - $T < 0 \Rightarrow$ sink. $T > 0 \Rightarrow$ source.
- $\mathbf{y}' = A\mathbf{y}$,

$$A = \begin{pmatrix} -2 & -18 & -7 & -14 \\ 1 & 6 & 2 & 5 \\ 2 & 2 & -3 & 0 \\ -2 & -8 & -1 & -6 \end{pmatrix}.$$

  - $A$ has eigenvalues $-1$, $-2$, & $-1 \pm i$.
  - $\mathbf{0}$ is asymptotically stable.

# Propostion

**Proposition:** Suppose that $A$ is an $n \times n$ matrix, $\lambda$ is a number, and $\mathbf{v}$ is a vector.

1. If $[A - \lambda I]\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}\mathbf{v}$.
2. If $[A - \lambda I]^2\mathbf{v} = \mathbf{0}$, then $e^{tA}\mathbf{v} = e^{\lambda t}(\mathbf{v} + t[A - \lambda I]\mathbf{v})$.
3. If $[A - \lambda I]^k\mathbf{v} = \mathbf{0}$, then

$$e^{tA}\mathbf{v} = e^{\lambda t}\left(\mathbf{v} + t[A - \lambda I]\mathbf{v} + \frac{t^2}{2!}[A - \lambda I]^2\mathbf{v} + \cdots \right.$$
$$\left. + \frac{t^{k-1}}{(k-1)!}[A - \lambda I]^{k-1}\mathbf{v}\right).$$