# Math 211

Lecture #32

Harmonic Motion

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## The Vibrating Spring

Newton's second law: $ma =$ total force.

- Forces acting:
  - Gravity $mg$.
  - Restoring force $R(x)$.
  - Damping force $D(v)$.
  - External force $F(t)$.
- Including all of the forces, Newton's law becomes

$$ma = mg + R(x) + D(v) + F(t)$$

Return



- Hooke's law: $R(x) = -kx$. $k > 0$ is the *spring constant*.
  - Spring-mass equilibrium $x_0 = mg/k$. Set $y = x - x_0$. Newton's law becomes

$$my'' = -ky + D(y') + F(t).$$

- Damping force $D(y') = -\mu y'$. $\mu \geq 0$ is the *damping constant.* Newton's law becomes

$$my'' = -ky - \mu y' + F(t), \quad \text{or}$$
$$my'' + \mu y' + ky = F(t), \quad \text{or}$$
$$y'' + \frac{\mu}{m} y' + \frac{k}{m} y = \frac{1}{m} F(t).$$

- This is the equation of the vibrating spring.

Return

## $RLC$ Circuit

$$LI'' + RI' + \frac{1}{C}I = E'(t), \quad \text{or}$$
$$I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t).$$

- This is the equation of the $RLC$ circuit.

Return Vibrating spring equation



## Harmonic Motion

- Spring: $y'' + \frac{\mu}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t)$.
- Circuit: $I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t)$.
- Essentially the same equation. Use

$$x'' + 2cx' + \omega_0^2 x = f(t).$$

  - We call this the equation for *harmonic motion.*
  - It includes both the vibrating spring and the $RLC$ circuit.

Return



## The Equation for Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = f(t).$$

- $\omega_0$ is the *natural frequency.*
  - Spring: $\omega_0 = \sqrt{k/m}$.
  - Circuit: $\omega_0 = \sqrt{1/LC}$.
- $c$ is the *damping constant.*
  - Spring: $2c = \mu/m$.
  - Circuit: $2c = R/L$.
- $f(t)$ is the *forcing term.*

Return

## Simple Harmonic Motion

No forcing, and no damping.

$$x'' + \omega_0^2 x = 0$$

- $p(\lambda) = \lambda^2 + \omega_0^2$, $\lambda = \pm i\omega_0$.
- Fundamental set of solutions: $x_1(t) = \cos\omega_0 t$ & $x_2(t) = \sin\omega_0 t$.
- General solution: $x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$.
- Every solution is periodic at the natural frequency $\omega_0$.
  - The period is $T = 2\pi/\omega_0$.

Return



## Amplitude and Phase

- Put $C_1$ and $C_2$ in polar coordinates:

$$C_1 = A\cos\phi, \ \& \ C_2 = A\sin\phi.$$

- Then $x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$

$$= A\cos(\omega_0 t - \phi).$$

- $A$ is the *amplitude*; $A = \sqrt{C_1^2 + C_2^2}$.
- $\phi$ is the *phase*; $\tan\phi = C_2/C_1$.

Return



## Examples

- $C_1 = 3$, $C_2 = 4 \ \Rightarrow \ A = 5$, $\phi = 0.9273$.
- $C_1 = -3$, $C_2 = 4 \ \Rightarrow \ A = 5$, $\phi = 2.2143$.
- $C_1 = -3$, $C_2 = -4 \ \Rightarrow \ A = 5$, $\phi = -2.2143$.

Return Amplitude & phase

## Example of Simple Harmonic Motion

$$x'' + 16x = 0,\ x(0) = -2\ \&\ x'(0) = 4$$

- Natural frequency: $\omega_0^2 = 16 \ \Rightarrow\ \omega_0 = 4$.
- General solution: $x(t) = C_1 \cos 4t + C_2 \sin 4t$.
- IC: $-2 = x(0) = C_1$, and $4 = x'(0) = 4C_2$.
- Solution

$$\begin{aligned} x(t) &= -2\cos 2t + \sin 2t \\ &= \sqrt{5}\cos(2t - 2.6779). \end{aligned}$$





## Damped Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = 0$$

- $p(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$; roots $-c \pm \sqrt{c^2 - \omega_0^2}$.
- Three cases
  - $c < \omega_0$ — *underdamped case*
  - $c > \omega_0$ — *overdamped case*
  - $c = \omega_0$ — *critically damped case*





## Underdamped Case

- $c < \omega_0$
- Two complex roots $\lambda$ and $\overline{\lambda}$, where $\lambda = -c + i\omega$ and $\omega = \sqrt{\omega_0^2 - c^2}$ .
- General solution

$$\begin{aligned} x(t) &= e^{-ct}[C_1 \cos \omega t + C_2 \sin \omega t] \\ &= Ae^{-ct}\cos(\omega t - \phi) \end{aligned}.$$

## Overdamped Case

- $c > \omega_0$ , so two real roots

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2}$$
$$\lambda_2 = -c + \sqrt{c^2 - \omega_0^2}.$$

- $\lambda_1 < \lambda_2 < 0.$
- General solution

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

Return



## Critically Damped Case

- $c = \omega_0$
- One negative real root $\lambda = -c$ with multiplicity 2.
- General solution

$$x(t) = e^{-ct}[C_1 + C_2 t].$$

Return



## Roots and Solutions

- If the characteristic polynomial has two distinct real roots $\lambda_1$ and $\lambda_2$, then $y(_1 t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$ are a fundamental set of solutions.
- If $\lambda$ is a root to the characteristic polynomial of multiplicity 2, then $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ are a fundamental set of solutions.
- If $\lambda = \alpha + i\beta$ is a complex root of the characteristic equation, then $z(t) = e^{\lambda t}$ and $\overline{z}(t) = e^{\overline{\lambda} t}$ are a complex valued fundamental set of solutions.
  - $x(t) = e^{\alpha t}\cos\beta t$ and $y(t) = e^{\alpha t}\sin\beta t$ are a real valued fundamental set of solutions.

Return