Math 211

Lecture #33

Inhomgeneous Equations
Forced Harmonic Motion

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Inhomogeneous Equations

$$y'' + py' + qy = f(t)$$

- The corresponding homogeneous equation is

$$y'' + py' + qy = 0$$

  - We know how to find a fundamental set of solutions y_1 and y_2.
  - The general solution of the homogeneous equation is $y_h(t) = C_1 y_1(t) + C_2 y_2(t)$.

Return

Theorem:

Assume

- $y_p(t)$ is a particular solution to the inhomogeneous equation $y'' + py' + qy = f(t)$;
- $y_1(t)$ & $y_2(t)$ is a fundamental set of solutions to the homogeneous equation $y'' + py' + qy = 0$.

Then the general solution to the inhomogeneous equation is

$$y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t).$$

We only need to find one particular solution.

Return Homogeneous equation

Method of Undetermined Coefficients

$$y'' + py' + qy = f(t)$$

The mantra for finding a particular solution is as follows:

- If the forcing term $f(t)$ has a form which is replicated under differentiation, then look for a particular solution of the same general form as the forcing term.

Return

Exponential Forcing Term

$$y'' + py' + qy = Ce^{at}$$

- Example: $y'' + 3y' + 2y = 4e^{-3t}$
- Try $y_p(t) = ae^{-3t}$; a to be determined.
  - Particular solution: $y_p(t) = 2e^{-3t}$.
- Homogeneous equation: $y'' + 3y' + 2y = 0$.
  - Fundamental set of solutions: e^{-2t} & e^{-t}.
- General solution to the inhomogeneous equation:

$$y(t) = 2e^{-3t} + C_1e^{-t} + C_2e^{-2t}.$$

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Trigonometric Forcing Term

$$y'' + py' + qy = A\cos\omega t + B\sin\omega t$$

- Example: $y'' + 4y' + 5y = 4\cos 2t - 3\sin 2t$
- Try $y_p(t) = a\cos 2t + b\sin 2t$
  - Particular solution: $y_p(t) = [28\cos 2t + 29\sin 2t]/65$.
- Homogeneous equation: $y'' + 4y' + 5y = 0$
  - Fund. set of sol'ns: $e^{-2t}\cos t$ & $e^{-2t}\sin t$.
- General solution to the inhomogeneous equation:

$$y(t) = \frac{28\cos 2t + 29\sin 2t}{65} + e^{-2t}[C_1\cos t + C_2\sin t].$$

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Complex Method

$$x'' + px' + qx = A\cos\omega t \quad \text{or}$$
$$y'' + py' + qy = A\sin\omega t.$$

- Solve $z'' + pz' + qz = Ae^{i\omega t}$. Try $z(t) = ae^{i\omega t}$.
  - $x_p(t) = \mathrm{Re}(z(t))$ and $y_p(t) = \mathrm{Im}(z(t))$.
- Example: $x'' + 4x' + 5x = 4\cos 2t$
  - Solve $z'' + 4z' + 5z = 4e^{2it}$. Try $z(t) = ae^{2it}$.
  - Particular solution: $z(t) = (4 - 32i)e^{2it}/65$.
- Particular solution to the real equation:

$$x_p(t) = \mathrm{Re}(z(t)) = [4\cos 2t + 32\sin 2t]/65.$$

Return

Polynomial Forcing Term

$$y'' + py' + qy = P(t)$$

- Example: $y'' - 3y' + 2y = 1 - 4t$.
  - Try $y(t) = a + bt$.
  - Particular solution: $y(t) = -5 - 2t$.
- General solution

$$y(t) = -5 - 2t + C_1e^t + C_2e^{2t}.$$

Return

Exceptional Cases

- Example: $y'' - 3y' + 2y = 3e^t$.
  - Try $y(t) = ae^t$
  - The method does not work because e^t is a solution to the associated homogeneous equation.
- Try $y(t) = ate^t$
  - Particular solution: $y_p(t) = -3te^t$.
- General solution: $y(t) = -3te^t + C_1e^t + C_2e^{2t}$.
- If the suggested particular solution does not work, multiply it by t and try again.

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Combination Forcing Term

Example $y'' + 5y' + 6y = 2e^{2t} - 5\cos t$

- Solve

$$y_1'' + 5y_1' + 6y_1 = 2e^{2t}$$
$$y_2'' + 5y_2' + 6y_2 = -5\cos t$$

- Set $y(t) = y_1(t) + y_2(t)$.

Forced Harmonic Motion

Assume an oscillatory forcing term:

$$y'' + 2cy' + \omega_0^2 y = A\cos\omega t$$

- A is the forcing amplitude
- ω is the forcing frequency
- ω_0 is the natural frequency.
- c is the damping constant.

Forced Undamped Harmonic Motion

$$y'' + \omega_0^2 y = A\cos\omega t$$

- Homogeneous equation: $y'' + \omega_0^2 y = 0$.
  - General solution: $y(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t$.
- $\omega \neq \omega_0$: Look for $x_p(t) = a\cos\omega t + b\sin\omega t$.
  - We find $x_p(t) = \dfrac{A}{\omega_0^2 - \omega^2}\cos\omega t$.
  - General solution:

$$x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{A}{\omega_0^2 - \omega^2}\cos\omega t.$$

- $\omega \neq \omega_0$ (cont.)
  - Initial conditions $x(0) = x'(0) = 0 \;\Rightarrow$

$$x(t) = \frac{A}{\omega_0^2 - \omega^2}[\cos\omega t - \cos\omega_0 t].$$

  - Example: $\omega_0 = 9,\ \omega = 8,\ A = \omega_0^2 - \omega^2 = 17.$
  - Set $\overline{\omega} = \dfrac{\omega_0 + \omega}{2}$ and $\delta = \dfrac{\omega_0 - \omega}{2}.$
  - Then $x(t) = \dfrac{A}{\omega_0^2 - \omega^2}[\cos\omega t - \cos\omega_0 t]$ $= \dfrac{A\sin\delta t}{2\overline{\omega}\delta}\sin\overline{\omega}t.$
  - Example: $\overline{\omega} = 8.5$ and $\delta = 0.5.$

- $\omega \neq \omega_0$ (cont.)

$$x(t) = \frac{A}{\omega_0^2 - \omega^2}[\cos\omega t - \cos\omega_0 t] = \frac{A\sin\delta t}{2\overline{\omega}\delta}\sin\overline{\omega}t.$$

  - The *envelope* $\pm\left|\dfrac{A\sin\delta t}{2\overline{\omega}\delta}\right|$ oscillates slowly with frequency δ.
  - The solution $x(t)$ shows a fast oscillation with frequency $\overline{\omega}$ and amplitude defined by the envelope.
  - This phenomenon is called *beats*. It occurs whenever two oscillations with frequencies that are close interfere.

- $\omega = \omega_0$

$$y'' + \omega_0^2 y = A\cos\omega_0 t.$$

  - We have an exceptional case. Try

$$x_p(t) = t[a\cos\omega t + b\sin\omega t].$$

  - We find

$$x_p(t) = \frac{A}{2\omega_0}t\sin\omega_0 t.$$

  - General solution

$$x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{A}{2\omega_0}t\sin\omega_0 t.$$

- $\omega = \omega_0$
  - Initial conditions $x(0) = x'(0) = 0 \Rightarrow$

$$x(t) = \frac{A}{2\omega_0} t \sin \omega_0 t.$$

    - Example: $\omega_0 = 5$, and $A = 2\omega_0 = 10$.

$$x(t) = t \sin 5t.$$

  - Oscillation with increasing amplitude.
  - First example of *resonance*.
    - Forcing at the natural frequency can cause oscillations that grow out of control.

Return

Roots and Solutions

- If the characteristic polynomial has two distinct real roots λ_1 and λ_2, then $y(_1t) = e^{\lambda_1 t}$ and $y_2(t) = e^{\lambda_2 t}$ are a fundamental set of solutions.
- If λ is a root to the characteristic polynomial of multiplicity 2, then $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ are a fundamental set of solutions.
- If $\lambda = \alpha + i\beta$ is a complex root of the characteristic equation, then $z(t) = e^{\lambda t}$ and $\overline{z}(t) = e^{\overline{\lambda} t}$ are a complex valued fundamental set of solutions.
  - $x(t) = e^{\alpha t} \cos \beta t$ and $y(t) = e^{\alpha t} \sin \beta t$ are a real valued fundamental set of solutions.

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