Math 211

Lecture #34

Forced Harmonic Motion

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Forced Harmonic Motion

Assume an oscillatory forcing term:

$$y'' + 2cy' + \omega_0^2 y = A\cos\omega t$$

- A is the forcing amplitude
- ω is the forcing frequency
- ω_0 is the natural frequency.
- c is the damping constant.

Forced Undamped Harmonic Motion

$$y'' + \omega_0^2 y = A\cos\omega t$$

- Homogeneous equation: $y'' + \omega_0^2 y = 0$.
  - General solution: $y(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t$.
- $\omega \neq \omega_0$: Look for $x_p(t) = a\cos\omega t + b\sin\omega t$.
  - We find $x_p(t) = \dfrac{A}{\omega_0^2 - \omega^2}\cos\omega t$.
  - General solution:

$$x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{A}{\omega_0^2 - \omega^2}\cos\omega t.$$

- $\omega \neq \omega_0$ (cont.)
  - Initial conditions $x(0) = x'(0) = 0 \;\Rightarrow$
  $$x(t) = \frac{A}{\omega_0^2 - \omega^2}[\cos\omega t - \cos\omega_0 t].$$
    - Example: $\omega_0 = 9,\ \omega = 8,\ A = \omega_0^2 - \omega^2 = 17.$
  - Set $\overline{\omega} = \dfrac{\omega_0 + \omega}{2}$ and $\delta = \dfrac{\omega_0 - \omega}{2}.$
  - Then $x(t) = \dfrac{A}{\omega_0^2 - \omega^2}[\cos\omega t - \cos\omega_0 t]$
  $$= \frac{A \sin\delta t}{2\overline{\omega}\delta} \sin\overline{\omega} t.$$
    - Example: $\overline{\omega} = 8.5$ and $\delta = 0.5.$

- $\omega \neq \omega_0$ (cont.)
$$x(t) = \frac{A}{\omega_0^2 - \omega^2}[\cos\omega t - \cos\omega_0 t] = \frac{A \sin\delta t}{2\overline{\omega}\delta} \sin\overline{\omega} t.$$
  - The *envelope* $\pm\left|\dfrac{A\sin\delta t}{2\overline{\omega}\delta}\right|$ oscillates slowly with frequency δ.
  - The solution $x(t)$ shows a fast oscillation with frequency $\overline{\omega}$ and amplitude defined by the envelope.
  - This phenomenon is called *beats*. It occurs whenever two oscillations with frequencies that are close interfere.

- $\omega = \omega_0$
$$y'' + \omega_0^2 y = A\cos\omega_0 t.$$
  - This is an exceptional case. Try
  $$x_p(t) = t[a\cos\omega t + b\sin\omega t].$$
  - We find
  $$x_p(t) = \frac{A}{2\omega_0} t \sin\omega_0 t.$$
  - General solution
  $$x(t) = C_1\cos\omega_0 t + C_2 \sin\omega_0 t + \frac{A}{2\omega_0} t\sin\omega_0 t.$$

- $\omega = \omega_0$
  - Initial conditions $x(0) = x'(0) = 0 \Rightarrow$

$$x(t) = \frac{A}{2\omega_0} t \sin \omega_0 t.$$

    - Example: $\omega_0 = 5$, and $A = 2\omega_0 = 10$.

$$x(t) = t \sin 5t.$$

  - Oscillation with increasing amplitude.
  - First example of *resonance*.
    - Forcing at the natural frequency can cause oscillations that grow out of control.

Return

Forced, Damped Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = A \cos \omega t$$

Use the complex method.

- Solve $z'' + 2cz' + \omega_0^2 z = Ae^{i\omega t}$.
- We try $z(t) = ae^{i\omega t}$ and get

$$\begin{aligned} z'' + 2cz' + \omega_0^2 z &= [(i\omega)^2 + 2c(i\omega) + \omega_0^2]ae^{i\omega t} \\ &= P(i\omega)z \end{aligned},$$

  where $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$ is the characteristic polynomial.
- The complex solution is $z(t) = \frac{1}{P(i\omega)} Ae^{i\omega t}$.
- The real solution is $x_p(t) = \mathrm{Re}(z(t))$.

Return

Example

$$x'' + 5x' + 4x = 50 \cos 3t$$

- $P(\lambda) = \lambda^2 + 5\lambda + 4$.
  - $P(i\omega) = P(3i) = -5 + 15i$
- $z(t) = \frac{1}{P(i\omega)} \cdot 50e^{3it}$

$$= -[(\cos 3t - 3\sin 3t) + i(\sin 3t + 3\cos 3t)]$$

- $x_p(t) = \mathrm{Re}(z(t)) = 3\sin 3t - \cos 3t$.

Return Particular solution

The Transfer Function

- The complex solution is

$$z(t) = \frac{1}{P(i\omega)} Ae^{i\omega t} = H(i\omega) Ae^{i\omega t},$$

where $H(i\omega) = \dfrac{1}{P(i\omega)}$ is called the *transfer function*.

- We will use complex polar coordinates to write

$$H(i\omega) = G(\omega) e^{-i\phi(\omega)},$$

where $G(\omega) = |H(i\omega)|$ is the called the *gain* and $\phi(\omega)$ is called the *phase shift*.

The Gain and Phase Shift

- If $P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$ is the characteristic polynomial, then $P(i\omega) = Re^{i\phi}$, where

$$R = \sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}, \quad \text{and}$$
$$\phi = \operatorname{arccot}\left(\frac{\omega_0^2 - \omega^2}{2c\omega}\right).$$

- The transfer function is

$$H(i\omega) = \frac{1}{P(i\omega)} = \frac{1}{R} e^{-i\phi} = G(\omega) e^{-i\phi}.$$

  - The gain $G(\omega) = \dfrac{1}{R} = \dfrac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}$.

- The complex particular solution is

$$\begin{aligned} z(t) &= H(i\omega) Ae^{i\omega t} = G(\omega) e^{-i\phi} \cdot Ae^{i\omega t} \\ &= G(\omega) Ae^{i(\omega t - \phi)}. \end{aligned}$$

- The real particular solution is

$$x_p(t) = \mathrm{Re}(z(t)) = G(\omega) A \cos(\omega t - \phi).$$

  - The amplitude of x_p is $G(\omega)A$, and the phase is ϕ.

- The general solution is

$$\begin{aligned} x(t) &= x_p(t) + x_h(t) \\ &= G(\omega)A\cos(\omega t - \phi) + x_h(t), \end{aligned}$$

where $x_h(t)$ is the general solution of the homogeneous equation.

- $x_h(t) \to 0$ as t increases, so x_h is called the *transient term*.
- $x_p(t) = G(\omega)A\cos(\omega t - \phi)$ is called the *steady-state solution*.

Example

$$x'' + 5x' + 4x = 50\cos 3t$$

- $G(\omega) = \dfrac{1}{\sqrt{(4-\omega^2)^2 + 25\omega^2}}$ and

$$\phi = \operatorname{arccot}\left(\frac{4-\omega^2}{5\omega}\right).$$

  - With $\omega = 3$,

$$\begin{aligned} G(3) &= \frac{1}{5\sqrt{10}} \approx 0.0632 \\ \phi &= \operatorname{arccot}(-3/5) \approx 2.1112. \end{aligned}$$

  - SS solution $x_p(t) = G(3)A\cos(3t - \phi)$.

The Steady-State Solution

$$x_p(t) = G(\omega)A\cos(\omega t - \phi).$$

- The forcing function is $A\cos\omega t$.
- Properties of the steady-state response:
  - It is oscillatory at the driving frequency.
  - The amplitude is the product of the gain, $G(\omega)$, and the amplitude of the forcing function.
  - It has a phase shift of ϕ with respect to the forcing function.

The Gain

$$G(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}$$

Set $\omega = s\omega_0$ and $c = D\omega_0/2$ (or $s = \omega/\omega_0$ and $D = 2c/\omega_0$). Then

$$G(\omega) = \frac{1}{\omega_0^2} \frac{1}{\sqrt{(1 - s^2)^2 + D^2 s^2}}$$