Math 211

Lecture #35

Nonlinear Systems

November 17, 2003

Interacting Species

- Two species with populations x_1 & x_2.
- Interaction between the species can be helpful or detrimental.
- Basic model

$$x_1' = r_1 x_1$$
$$x_2' = r_2 x_2$$

  - r_1 & r_2 are the *reproductive rates*.

Return

Reproductive Rates

- If $x_2 = 0$ the reproductive rate for x_1 is

$$r_1 = a_1 - b_1 x_1.$$

  - $a_1 > 0 \Rightarrow$ natural growth.
  - $a_1 < 0 \Rightarrow$ natural decline.
  - $b_1 = 0$ Malthusian growth.
  - $b_1 > 0$ logistic growth.

Return

- If $x_2 > 0$ the reproductive rate for x_1 is

$$r_1 = a_1 - b_1 x_1 + c_1 x_2.$$

  - $c_1 > 0 \Rightarrow$ interaction is helpful to x_1.
  - $c_1 < 0 \Rightarrow$ interaction is detrimental to x_1.
- The reproductive rate for x_2 is

$$r_2 = a_2 - b_2 x_2 + c_2 x_1.$$

- The model for interacting species is

$$x_1' = (a_1 - b_1 x_1 + c_1 x_2) x_1$$
$$x_2' = (a_2 - b_2 x_2 + c_2 x_1) x_2$$

Return

Predator Prey Model

Rabbits & foxes, fish & sharks, and cottony cushion scale insect & ladybird beetle.

- F = fish & S = sharks.

$$F' = (a - bS)F$$
$$S' = (-c + dF)S$$

or

$$F' = (a - eF - bS)F$$
$$S' = (-c + dF)S$$

$a = 3,\ b = 3,\ c = 1,\ d = 3,\ e = 3.$

Return Interacting species Reproductive rates

Competing Species

Cattle and sheep.

- x_1 and x_2 competing for resources.

$$x_1' = (a_1 - b_1 x_1 + c_1 x_2) x_1$$
$$x_2' = (a_2 - b_2 x_2 + c_2 x_1) x_2$$

  - $a_i > 0$, $b_i > 0$, & $c_i < 0$
- Example:

$$x' = (5 - 2x - y)x$$
$$y' = (7 - 2x - 3y)y$$

Return

Linearization

The principal idea of differential calculus:

- Approximate nonlinear mathematical objects by linear ones.
- Example: Approximate the function $f(y)$ near y_0 by a linear function.

$$f(y_0 + h) = f(y_0) + f'(y_0)h + R(h)$$

where $\lim_{h\to 0} \frac{R(h)}{h} = 0.$

- ♦ The linear function is $L(h) = f(y_0) + f'(y_0)h$.

Linearization of an ODE at an Equilibrium Point

$$y' = f(y)$$

- Assume $f(y_0) = 0$ and $f'(y_0) \neq 0$.
- Set $y = y_0 + u$. Get

$$u' = f(y_0 + u) = f'(y_0)u + R(u)$$

- Approximate by the linear differential equation

$$\tilde{u}' = f'(y_0)\tilde{u}$$

- If $f'(y_0) \neq 0$ the equilibrium point of the linearization at 0 has the same stability properties as that of the nonlinear equation at y_0.

Linearization of a Planar System

$$x' = f(x, y)$$
$$y' = g(x, y)$$

- Assume (x_0, y_0) is an equilibrium point, so

$$f(x_0, y_0) = g(x_0, y_0) = 0$$

- We have by Taylor's theorem

$$f(x_0 + u, y_0 + v) = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v)$$

$$g(x_0 + u, y_0 + v) = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v)$$

where $\frac{R_f(u,v)}{\sqrt{u^2+v^2}} \to 0$ and $\frac{R_g(u,v)}{\sqrt{u^2+v^2}} \to 0.$

Linearization at (x_0, y_0)

- Set $x = x_0 + u$ and $y = y_0 + v$. The system becomes

$$u' = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v)$$
$$v' = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v)$$

- Approximate by the linear system

$$\tilde{u}' = \frac{\partial f}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0)\tilde{v}$$
$$\tilde{v}' = \frac{\partial g}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0)\tilde{v}$$

Matrix Form of the Linearization

- Set $\mathbf{u} = (\tilde{u}, \tilde{v})^T$ and introduce the *Jacobian matrix*

$$J = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

  The linearization becomes

$$\mathbf{u}' = J\mathbf{u}.$$

- We can solve the linear system explicitly.
- Does it give information about the original nonlinear system?

Theorem: Consider the planar system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

where f and g are continuously differentiable. Suppose that (x_0, y_0) is an equilibrium point. If the linearization at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

Generic Equilibrium Points

- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - All occupy large open subsets of the trace-determinant plane.
- Nongeneric types
  - Center and others. Occupy pieces of the boundaries.

Examples

- Predator prey
- Competing species
- Center

$$x' = y + \alpha x(x^2 + y^2)$$
$$y' = -x + \alpha y(x^2 + y^2)$$