Math 211

Lecture #36

The Use of the Linearization

November 19, 2003

Linearization of a Planar System

$$x' = f(x, y)$$
$$y' = g(x, y)$$

- Assume (x_0, y_0) is an equilibrium point, so

$$f(x_0, y_0) = g(x_0, y_0) = 0$$

- We have by Taylor's theorem

$$f(x_0 + u, y_0 + v) = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v)$$
$$g(x_0 + u, y_0 + v) = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v)$$

where $\frac{R_f(u,v)}{\sqrt{u^2+v^2}} \to 0$ and $\frac{R_g(u,v)}{\sqrt{u^2+v^2}} \to 0$.

Return

Linearization at (x_0, y_0)

- Set $x = x_0 + u$ and $y = y_0 + v$. The system becomes

$$u' = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v)$$
$$v' = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v)$$

- Approximate by the linear system

$$\tilde{u}' = \frac{\partial f}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0)\tilde{v}$$
$$\tilde{v}' = \frac{\partial g}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0)\tilde{v}$$

Return

Matrix Form of the Linearization

- Set $\mathbf{u} = (\tilde{u}, \tilde{v})^T$ and introduce the *Jacobian matrix*

$$J = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

  The linearization becomes

$$\mathbf{u}' = J\mathbf{u}.$$

- We can solve the linear system explicitly.
- The linearization gives us information about the original system.

Theorem 1

Theorem: Consider the planar system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

where f and g are continuously differentiable. Suppose that (x_0, y_0) is an equilibrium point. If the linearization at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

- Generic types: Saddle, nodal source, nodal sink, spiral source, and spiral sink. — All occupy large open subsets of the trace-determinant plane.
- Nongeneric types: Center and others. — Occupy pieces of the boundaries between the generic points.

Examples

- Center.

$$x' = y + \alpha x(x^2 + y^2)$$
$$y' = -x + \alpha y(x^2 + y^2)$$

  - $\alpha > 0 \Rightarrow (0,0)^T$ is unstable.
  - $\alpha < 0 \Rightarrow (0,0)^T$ is a sink.
- Competing species.

$$x' = (5 - 2x - y)x$$
$$y' = (7 - 2x - 3y)y$$

- Default system in pplane.

Linear Analysis of Equilibrium Points

- Provides a good qualitative picture of how solutions behave near generic equilibrium points.
- Provides limited qualitative information about the solutions near nongeneric equilibrium points.
  - A linear center could be a spiral source or a spiral sink.
- Provides no information about the global behavior of solutions to nonlinear systems.

Higher Dimensional Systems

Autonomous equation $\mathbf{y}' = \mathbf{f}(\mathbf{y})$.

- $\mathbf{y} = (y_1, y_2, \cdots, y_n)^T$, $\mathbf{y}_0$ is an equilibrium point.
- $\mathbf{f}(\mathbf{y}) = (f_1(\mathbf{y}), f_2(\mathbf{y}), \cdots, f_n(\mathbf{y}))^T$
- J is the Jacobian matrix
- $\mathbf{f}(\mathbf{y}_0 + \mathbf{u}) = J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u})$ where $\lim_{\mathbf{u}\to\mathbf{0}} \frac{\mathbf{R}(\mathbf{u})}{|\mathbf{u}|} = \mathbf{0}$.
- Set $\mathbf{y} = \mathbf{y}_0 + \mathbf{u}$. The system becomes

$$\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u}).$$

- The linearization is $\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u}$.

Theorem 2

Theorem: Suppose that $\mathbf{y}_0$ is an equilibrium point for $\mathbf{y}' = \mathbf{f}(\mathbf{y})$. Let J be the Jacobian of $\mathbf{f}$ at $\mathbf{y}_0$.

1. Suppose that the real part of every eigenvalue of J is negative. Then $\mathbf{y}_0$ is an asymptotically stable equilibrium point.
2. Suppose that J has at least one eigenvalue with positive real part. Then $\mathbf{y}_0$ is an unstable equilibrium point.

Example

$$x' = -2x - 4y + 2xy$$
$$y' = x - 6y + x^2 - y^2$$

- The origin $(0,0)$ is an equilibrium point.
- The Jacobian has one eigenvalue, $\lambda = -4$, of algebraic multiplicity 2.
- First theorem does not apply.
- Second theorem $\Rightarrow$ the origin is a sink.

The Lorenz System

$$x' = -ax + ay$$
$$y' = rx - y - xz$$
$$z' = -bz + xy$$

- Use $a = 10$, $b = 8/3$, and $r = 5, 20, 28, 100$.