

# Math 211

Lecture #38

Invariant Sets

November 24, 2003

## Review of Methods

Linearization at an equilibrium point

- $\mathbf{y}' = \mathbf{f}(\mathbf{y})$  has an equilibrium point at  $\mathbf{y}_0$ .
- The linearization  $\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u}$  has an equilibrium point at  $\mathbf{u} = \mathbf{0}$ .
- The linearization can sometimes predict the behavior of solutions to the nonlinear system *near the equilibrium point*.
  - ♦ The linearization gives only local information.
  - ♦ We need ways to discover the global behavior of solutions.

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## Invariant Sets

**Definition:** A set  $S$  is (positively) invariant for the system  $\mathbf{y}' = \mathbf{f}(\mathbf{y})$  if  $\mathbf{y}(0) = \mathbf{y}_0 \in S$  implies that  $\mathbf{y}(t) \in S$  for all  $t \geq 0$ .

- Examples:
  - ♦ An equilibrium point.
  - ♦ Any solution curve.

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### Example — Competing Species

$$x' = (5 - 2x - y)x$$

$$y' = (7 - 2x - 3y)y$$

- The positive  $x$ - and  $y$ -axes are invariant.
- The positive quadrant is invariant.
  - ♦ Populations should remain nonnegative.
- The set  $S = \{(x, y) \mid 0 < x < 3, 0 < y < 3\}$  is positively invariant.

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### Nullclines

$$x' = f(x, y)$$

$$y' = g(x, y)$$

**Definition:** The  $x$ -nullcline is the set defined by  $f(x, y) = 0$ .  
The  $y$ -nullcline is the set defined by  $g(x, y) = 0$ .

- Along the  $x$ -nullcline the vector field points up or down.
- Along the  $y$ -nullcline the vector field points left or right.
- The nullclines intersect at the equilibrium points.

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### Competing Species

$$x' = (5 - 2x - y)x$$

$$y' = (7 - 2x - 3y)y$$

- $x$ -nullcline: two lines  $x = 0$  and  $2x + y = 5$ .
- $y$ -nullcline: two lines  $y = 0$  and  $2x + 3y = 7$ .
- Two of the four regions in the positive quadrant defined by the nullclines are positively invariant.
- This information allows us to predict that all solutions in the positive quadrant  $\rightarrow (2, 1)$  as  $t \rightarrow \infty$ .

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[Nullclines](#)

## Competing Species – 2<sup>nd</sup> Example

$$x' = (1 - x - y)x$$

$$y' = (4 - 7x - 3y)y$$

- The axes are invariant. The positive quadrant is invariant.
- The equilibrium point at  $(1/4, 3/4)$  is a saddle point.
- Almost all solutions go to one of the nodal sinks  $(0, 4/3)$  or  $(1, 0)$ .

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**Definition:** The *basin of attraction* of a sink  $y_0$  consists of all points  $y$  such that the solution starting at  $y$  approaches  $y_0$  as  $t \rightarrow \infty$ .

- In the example, the basins of attraction of the two sinks are separated by the stable orbits of the saddle point.
- The stable and unstable orbits of a saddle point are called *separatrices*. (Separatrices is the plural of separatrix.)

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## Summary

- Sometimes the understanding of invariant sets can help us understand the long term behavior of all solutions.
- Nullclines can sometimes help us find informative invariant sets.
- None of this helps us understand the predator-prey system.