

Math 211

Lecture #39

Long Term Behavior of Planar Systems

November 26, 2003

Example

$$x' = (2 - x - y)x$$

$$y' = (3 - x^2 - y^2)y$$

- The axes are invariant. Each quadrant is invariant.
- x -nullcline: two lines $x = 0$ and $x + y = 2$.
- y -nullcline: line $y = 0$ and circle $x^2 + y^2 = 3$.
- The equilibrium points are $(0, 0)$, $(0, \sqrt{3})$, $(0, -\sqrt{3})$, $(2, 0)$, (a, b) , and (b, a) , where $a = 1 - 1/\sqrt{2}$ and $b = 2 - a = 1 + 1/\sqrt{2}$.

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Example (cont.)

- Classification: $(0, 0)$ is a nodal source, $(0, \sqrt{3})$, $(0, -\sqrt{3})$, and (b, a) are saddles, $(2, 0)$ and (a, b) are nodal sinks.
- The nullclines divide the positive quadrant into 5 regions, of which 3 are invariant.
- Almost all solution curves in the positive quadrant are attracted to the nodal sinks at $(2, 0)$ and (a, b) .
- The basins of attraction of $(2, 0)$ and (a, b) are separated by the stable solution curves for the saddle at (b, a) .

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Example

Basic Question about a System $y' = f(y)$

- What happens to all solutions as $t \rightarrow \infty$?
- What are the possibilities as $t \rightarrow \infty$?
 - ♦ Is there a small list of all possibilities?
 - ♦ We need a definitive notion of what a "possibility" is.

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Limit Sets

Definition: The (forward) limit set of the solution $y(t)$ that starts at y_0 is the set of all limit points of the solution curve. It is denoted by $\omega(y_0)$.

- $x \in \omega(y_0)$ if there is a sequence $t_k \rightarrow \infty$ such that $y(t_k) \rightarrow x$.
- What kinds of sets can be limit sets?
 - ♦ The empty set.
 - ♦ Equilibrium points.
 - ♦ Periodic solution curves. Including limit cycles.
 - ♦ Strange attractors in $d \geq 3$.
- Is there a small list of all possible limit sets?

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[Possibility](#)

Properties of Limit Sets

Theorem: Suppose that the system $y' = f(y)$ is defined in the set U .

1. If the solution curve starting at y_0 stays in a bounded subset of U , then the limit set $\omega(y_0)$ is not empty.
2. Any limit set is both positively and negatively invariant.

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