Math 211

Lecture #39

Long Term Behavior of Planar Systems

November 26, 2003

Example

$$x' = (2 - x - y)x$$
$$y' = (3 - x^2 - y^2)y$$

- The axes are invariant. Each quadrant is invariant.
- x-nullcline: two lines $x = 0$ and $x + y = 2$.
- y-nullcline: line $y = 0$ and circle $x^2 + y^2 = 3$.
- The equilibrium points are $(0, 0)$, $(0, \sqrt{3})$, $(0, -\sqrt{3})$, $(2, 0)$, (a, b), and (b, a), where $a = 1 - 1/\sqrt{2}$ and $b = 2 - a = 1 + 1/\sqrt{2}$.

Example (cont.)

- Classification: $(0,0)$ is a nodal source, $(0,\sqrt{3})$, $(0,-\sqrt{3})$, and (b,a) are saddles, $(2,0)$ and (a,b) are nodal sinks.
- The nullclines divide the postive quadrant into 5 regions, of which 3 are invariant.
- Almost all solution curves in the positive quadrant are attracted to the nodal sinks at $(2,0)$ and (a,b) .
- The basins of attraction of $(2,0)$ and (a,b) are separated by the stable solution curves for the saddle at (b,a).

Basic Question about a System $\mathbf{y}' = \mathbf{f}(\mathbf{y})$

- What happens to all solutions as $t \to \infty$?
- What are the possibilities as $t \to \infty$?
  - Is there a small list of all possibilities?
  - We need a definitive notion of what a "possibility" is.

Limit Sets

Definition: The (forward) limit set of the solution $\mathbf{y}(t)$ that starts at $\mathbf{y}_0$ is the set of all limit points of the solution curve. It is denoted by $\omega(\mathbf{y}_0)$.

- $\mathbf{x} \in \omega(\mathbf{y}_0)$ if there is a sequence $t_k \to \infty$ such that $\mathbf{y}(t_k) \to \mathbf{x}$.
- What kinds of sets can be limit sets?
  - The empty set.
  - Equilibrium points.
  - Periodic solution curves. Including limit cycles.
  - Strange attractors in $d \geq 3$.
- Is there a small list of all possibile limit sets?

Properties of Limit Sets

Theorem: Suppose that the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ is defined in the set U.

1. If the solution curve starting at $\mathbf{y}_0$ stays in a bounded subset of U, then the limit set $\omega(\mathbf{y}_0)$ is not empty.
2. Any limit set is both positively and negatively invariant.

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