Math 211

Lecture #40

Limits Sets of Solution Curves

December 1, 2003

Basic Question about $\mathbf{y}' = \mathbf{f}(\mathbf{y})$

- The (forward) limit set of the solution $\mathbf{y}(t)$ that starts at $\mathbf{y}_0$ is the set of all limit points of the solution curve. It is denoted by $\omega(\mathbf{y}_0)$.
  - $\mathbf{x} \in \omega(\mathbf{y}_0)$ if there is a sequence $t_k \to \infty$ such that $\mathbf{y}(t_k) \to \mathbf{x}$.
- What is $\omega(\mathbf{y}_0)$ for all $\mathbf{y}_0$?
- Examples:
  - The empty set.
  - Equilibrium points.
  - Periodic solution curves. Including limit cycles.
  - Strange attractors in $d \geq 3$.

Return

Properties of Limit Sets

Theorem: Suppose that the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ is defined in the set U.

1. If the solution curve starting at $\mathbf{y}_0$ stays in a bounded subset of U, then the limit set $\omega(\mathbf{y}_0)$ is not empty.
2. Any limit set is both positively and negatively invariant.

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Example

$$x' = 5y + x(9 - x^2 - y^2)$$
$$y' = -5x + y(9 - x^2 - y^2)$$

- The origin is a spiral source.
- In polar coordinates the system is

$$r' = r(9 - r^2)$$
$$\theta' = -5$$

- All solution curves approach the circle $x^2 + y^2 = 9$.
  - The circle $x^2 + y^2 = 9$ is a solution curve.

Return Definition

Limit Cycle

Definition: A limit cycle is a closed solution curve which is the limit set of nearby solution curves. If the solution curves spiral into the limit cycle as $t \to \infty$, it is a attracting limit cycle. If they spiral into the limit cycle as $t \to -\infty$, it is a repelling limit cycle.

- In the example the circle $x^2 + y^2 = 9$ is an attracting limit cycle.

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Types of Limit Set

- A limit cycle is a new type of phenomenon. However, the limit set is a periodic orbit, so the type of limit set is not new.
- We still have only two types of non-empty limits sets.
  - An equilibrium point.
  - A closed solution curve.
    - Periodic solutions.
    - Limit cycles.

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Example

$$x' = (x+1)(x+2y)(1-(x+y-1)/5)$$
$$y' = -(y+1)(2x+y)$$

- The lines $x = -1$ and $y = -1$ are invariant. The line $x + y = 1$ is invariant. The triangle is invariant.
- The vertices of the triangle are saddle points. The sides are separatrices.
- The origin is a spiral source.
- The limit set of any solution that starts in the triangle is the boundary of the triangle. This is a new type.

Planar Graph

Definition: A *planar graph* is a collection of points, called *vertices*, and non-intersecting curves, called *edges*, which connect the vertices. If the edges each have a direction the graph is said to be *directed*.

- The boundary of the triangle in the example is a directed planar graph.
- Look at Exercises 14 – 22 in Section 10.4.

Theorem: If S is a nonempty limit set of a solution of a planar system defined in a set $U \subset \mathbf{R}^2$, then S is one of the following:

- An equilibrium point.
- A closed solution curve.
- A directed planar graph with vertices that are equilibrium points, and edges which are solution curves.

These are called the *Poincaré-Bendixson alternatives.*

- Closed solution curves could be limit cycles.

Poincaré-Bendixson Theorem

Theorem: Suppose that R is a closed and bounded planar region that is positively invariant for a planar system. If R contains no equilibrium points, then there is a closed solution curve in R.

- The theorem is also true if the set R is negatively invariant.
- The closed solution curve might be a limit cycle.

Return Poincaré-Bendixson alternatives

Examples

1. $x' = x + y - x(x^2 + 3y^2)$
   $y' = -x + y - 2y^3$

- The set $\{(x, y) \,|\, 0.5 \le x^2 + y^2 \le 1\}$ is positively invariant. By the Poincaré-Bendixson theorem there is a limit cycle.

2. Rayleigh's example: $z'' + \mu z'[(z')^2 - 1] + z = 0$.

- There is a limit cycle.