Math 211

Lecture #41

The Pendulum

Predator-Prey

December 3, 2003

The Pendulum

- The angle θ satisfies the nonlinear differential equation

$$mL\theta'' = -mg\,\sin\theta - D\,\theta',$$

  - We will write this as

$$\theta'' + \mu\,\theta' + b\,\sin\theta = 0.$$

- Introduce $\omega = \theta'$ to get the system

$$\begin{aligned}\theta' &= \omega\\ \omega' &= -b\,\sin\theta - \mu\,\omega\end{aligned}$$

Analysis

- The equilibrium points are $(k\,\pi, 0)^T$ where k is any integer.
  - If k is odd the equilibrium point $(k\,\pi, 0)^T$ is a saddle.
  - If k is even the equilibrium point $(k\,\pi, 0)^T$ is
    - a center if $\mu = 0$ or
    - a sink if $\mu > 0$.

The Inverted Pendulum

- The angle θ measured from straight up satisfies the nonlinear differential equation

$$mL\theta'' = mg\,\sin\theta - D\,\theta',$$

or

$$\theta'' + \frac{D}{mL}\theta' - \frac{g}{L}\sin\theta = 0.$$

  - ♦ We will write this as

$$\theta'' + \mu\,\theta' - b\,\sin\theta = 0.$$

The Inverted Pendulum System

- Introduce $\omega = \theta'$ to get the system

$$\theta' = \omega$$

$$\omega' = b \sin\theta - \mu\,\omega$$

- The equilibrium point at $(0,0)^T$ is a saddle point and unstable.
- Can we find an automatic way of sensing the departure of the system from $(0,0)^T$ and moving the pivot to bring the system back to the unstable point at $(0,0)^T$?
  - Experimentally the answer is yes.

The Control System

- If we apply a force v moving the pivot to the right or left, then θ satisfies

$$mL\theta'' = mg\,\sin\theta - D\,\theta' - v\,\cos\theta,$$

- The system becomes

$$\theta' = \omega$$

$$\omega' = b\,\sin\theta - \mu\,\omega - u\,\cos\theta,$$

  where $u = v/mL$.

- Assume the force is a linear response to the detected value of θ, so $u = c\theta$, where c is a constant.

The Controlled System

- The Jacobian at the origin is

$$J = \begin{pmatrix} 0 & 1 \\ b - c & -\mu \end{pmatrix}$$

- The origin is asymptotically stable if $T = -\mu < 0$ and $D = c - b > 0$. Therefore require

$$c > b = \frac{g}{L}.$$

Predator-Prey

Lotka-Volterra system

$$x' = (a - by)x \quad \text{(prey – fish)}$$

$$y' = (-c + dx)y \quad \text{(predator – sharks)}$$

- Equilbrium points: $(0, 0)$ is a saddle, $(x_0, y_0) = (c/d, a/b)$ is a linear center.
- The axes are invariant.
- The positive quadrant is invariant.
- The solution curves appear to be closed. Is this actually true?

Solutions are Periodic

Along the solution curve $y = y(x)$ we have

$$\frac{dy}{dx} = \frac{y(-c + dx)}{x(a - by)}.$$

The solution is

$$H(x, y) = by - a \ln y + dx - c \ln x = C$$

- This is an implicit equation for the solution curve in the phase plane. $\Rightarrow$ All solution curves are closed, and represent periodic solutions.

Why Fishing Leads to More Fish

Compute the average of the fish & shark populations.

$$\frac{d}{dt}\ln x(t) = \frac{x'}{x} = a - by$$

$$0 = \frac{1}{T}\int_0^T \frac{d}{dt}\ln x(t)\,dt = a - b\overline{y}.$$

So $\overline{y} = a/b = y_0$. Similarly $\overline{x} = x_0 = c/d$.

The effect of fishing that does not distinquish between fish and sharks is the system

$$x' = (a - by)x - ex$$

$$y' = (-c + dx)y - ey$$

- This is the same system with a replaced by $a - e$ and c replaced by $c + e$.
- The average populations are

$$\overline{x}_1 = \frac{c + e}{d} \quad \text{and} \quad \overline{y}_1 = \frac{a - e}{b}$$

- Fishing causes the average fish population to increase and the average shark population to decrease.

Return

Cottony Cushion Scale Insect & the Ladybird Beetle

- The cottony cushion scale insect was accidentally introduced from Australia in 1868.
  - Threatened the citrus industry.
- In Australia the ladybird beetle is a natural predator on the insect and reduces the insects to a manageable level.
  - It was imported from Australia to help here.
- DDT kills the scale insect, so massive spraying was ordered.
  - Despite the warnings of mathematicians and biologists.
- The scale insect increased in numbers, as predicted by Volterra.