Math 211

Lecture #2

Separable Equations

Interval of Existence

The largest interval over which a solution can exist.

- Example: $y' = 1 + y^2 \quad \text{with} \quad y(0) = 1$
  - ◇ General solution: $y(t) = \tan(t + C)$
  - ◇ Initial Condition: $y(0) = 1 \Leftrightarrow C = \pi/4$.
- Solution: $y(t) = \tan(t + \pi/4)$ exists and is continuous for $-\pi/2 < t + \pi/4 < \pi/2$ or for $-3\pi/4 < t < \pi/4$.

Geometric Interpretation of

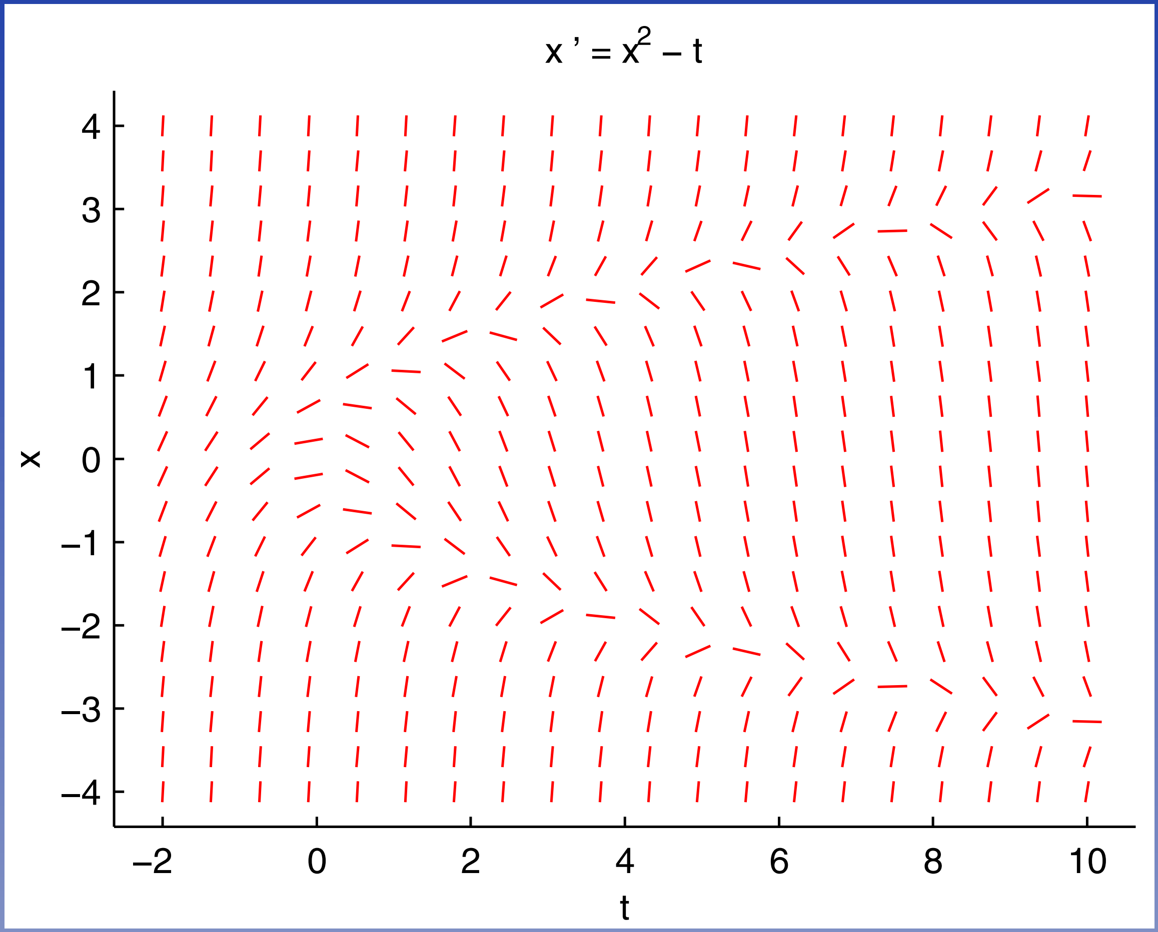
$$y' = f(t, y)$$

If $y(t)$ is a solution, and $y(t_0) = y_0$, then

$$y'(t_0) = f(t_0, y(t_0)) = f(t_0, y_0).$$

- The slope to the graph of $y(t)$ at the point (t_0, y_0) is given by $f(t_0, y_0)$.
- Imagine a small line segment attached to each point of the (t, y) plane with the slope $f(t, y)$.

The Direction Field



Autonomous Equations

General equation:

$$\frac{dy}{dt} = f(t, y) \qquad \frac{dy}{dt} = t - y^2$$

Autonomous equation:

$$\frac{dy}{dt} = f(y) \qquad \frac{dy}{dt} = y(1 - y)$$

In an *autonomous equation* the right hand side has no explicit dependence on the independent variable.

Equilibrium Points

Autonomous equation:

$$\frac{dy}{dt} = f(y) \qquad \frac{dy}{dt} = y(1-y)$$

- *Equilibrium point:*

$$f(y_0) = 0 \quad y_0 = 0 \quad \text{or} \quad 1$$

- *Equilibrium solution:*

$$y(t) = y_0 \quad y(t) = 0 \quad \text{and} \quad y(t) = 1$$

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Between Equilibrium Points

- $\frac{dy}{dt} = f(y) > 0 \Rightarrow\ y(t)$ is increasing.
- $\frac{dy}{dt} = f(y) < 0 \Rightarrow\ y(t)$ is decreasing.

Example: $\frac{dy}{dt} = y(1 - y)$

Separable Equations

General equation:

$$\frac{dy}{dt} = f(t, y) \qquad \frac{dy}{dt} = t - y^2$$

Separable equation:

$$\frac{dy}{dt} = g(y)h(t) \qquad \frac{dy}{dt} = t \sec y$$

In a *separable equation* the right hand side is a product of a function of the independent variable (t) and a function of the unknown function (y).

- Autonomous equations are separable.

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Solving Separable Equations

$$\frac{dy}{dt} = t \sec y$$

- Separate the variables:

$$\frac{dy}{\sec y} = t\,dt \quad \text{or} \quad \cos y\,dy = t\,dt$$

We have to worry about dividing by 0, but $\sec y$ is never equal to 0.

Integrate both sides

$$\int \cos y \, dy = \int t \, dt$$

$$\sin(y) + C_1 = \frac{1}{2}t^2 + C_2 \quad \text{or}$$

$$\sin(y) = \frac{1}{2}t^2 + C$$

where $C = C_1 - C_2$.

Solve for y

$$\sin(y) = \frac{1}{2}t^2 + C$$

$$y(t) = \arcsin\left(C + \frac{1}{2}t^2\right).$$

This is the general solution to $\dfrac{dy}{dt} = t\sec y$.

Solving Separable Equations

The three step solution process:

$$\frac{dy}{dt} = g(y)h(t)$$

- Separate the variables. $\displaystyle \frac{dy}{g(y)} = h(t)\,dt$
- Integrate both sides. $\displaystyle \int \frac{dy}{g(y)} = \int h(t)\,dt$
- Solve for y.

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Examples

- $y' = ry$
- $R' = \dfrac{\sin t}{1 + R}$ with $R(0) = 1, -2, -1$
- $x' = \dfrac{3t^2 x}{1 + 2x^2}$ with $x(0) = 1, 0$
- $y' = 1 + y^2$ with $y(0) = -1, 0, 1$

Why It Works

$$\frac{dy}{dt} = g(y)h(t)$$

$$\frac{1}{g(y)}\frac{dy}{dt} = h(t) \qquad \text{if } g(y) \neq 0$$

$$\int \frac{1}{g(y)}\frac{dy}{dt}\,dt = \int h(t)\,dt$$

$$\int \frac{1}{g(y)}\,dy = \int h(t)\,dt$$