Math 211

Lecture #5

Linear Equations

January 26, 2001

Linear Equations

$$x' = a(t)x + f(t), \quad \text{e.g.} \quad x' = \tan(t)x + 3\sin^2(t)$$

- The unknown function x and its derivative must appear *linearly*.
- The equation is *homogeneous* if $f = 0$
  - $x' = a(t)x, \quad \text{e.g.} \quad x' = \tan(t)x$
- The equation is *inhomogeneous* if $f \neq 0$

Homogenous Linear Equations

- Homogeneous linear equations are separable.

$$
\begin{aligned}
\frac{dx}{dt} &= a(t)x \\
\frac{dx}{x} &= a(t)\,dt \\
\ln|x(t)| &= \int a(t)\,dt \\
x(t) &= Ae^{\int a(t)\,dt}
\end{aligned}
$$

Return

Example: $x' = \tan(t)x$.

$$
\begin{aligned}
x(t) &= Ae^{\int \tan(t)\,dt} \\
&= Ae^{-\ln(\cos(t))} \\
&= \frac{A}{\cos t} \\
&= A\sec t
\end{aligned}
$$

Example: $x' = \tan(t)x + 3\sin^2(t)$

- Rewrite as $\quad x' - \tan(t)x = 3\sin^2(t)$
- Multiply by $\cos t$.

$$\cos(t)x' - \sin(t)x = 3\sin^2(t)\cos(t)$$

The left hand side is the derivative of $\cos(t)x$. So

$$[\cos(t)x]' = 3\sin^2(t)\cos(t)$$

- Integrate

$$\cos(t)x(t) = 3\int \sin^2(t)\cos(t)\,dt = \sin^3(t) + C$$

- Solve for x

$$x(t) = \frac{\sin^3(t) + C}{\cos(t)}.$$

How did we do that? Can we do it in general?

The Key Step for $x' = ax + f$

- Rewrite as $x' - ax = f$.
- Multiply by a function $u(t)$ so that

$$u[x' - ax] = [ux]'$$

$$ux' - aux = ux' + u'x$$

◇ True if $u' = -au$. Linear, homogeneous

$$u(t) = e^{-\int a(t)\,dt} \quad \text{is one solution.}$$

◇ u is called an *integrating factor.*

Solving $x' = a(t)x + f(t)$

- Rewrite as $x' - ax = f$.
- Multiply by the integrating factor

$$u(t) = e^{-\int a(t)\, dt}.$$

  - ◇ Makes the LHS an exact derivative
    $[ux]' = ux' - aux = uf$.
- Integrate: $u(t)x(t) = \int u(t)f(t)\, dt + C$.
- Solve for $x(t)$.

Examples

- $x' = -4x + 8, \quad x(0) = 0.$
- $x' = 2tx + e^{t^2}, \quad x(0) = 1.$
- $y' = 3y - t, \quad y(0) = 2.$
- $z' = (z+1)\cos t, \quad z(\pi) = -1.$