Math 211

Lecture #6

Mixing Problems

January 29, 2001

Solving $x' = a(t)x + f(t)$

- Rewrite as $x' - ax = f$.
- Multiply by the integrating factor
$$u(t) = e^{-\int a(t)\,dt}.$$
  - ◇ Makes the LHS an exact derivative $[ux]' = ux' - aux = uf$.
- Integrate: $u(t)x(t) = \int u(t)f(t)\,dt + C$.
- Solve for $x(t)$.

Return

Mixing Problem 1

A tank originally holds 500 gallons of pure water. At $t = 0$ there starts a flow of sugar water into the tank with a concentration of $\frac{1}{2}$ lbs/gal at a rate of 5 gal/min. There is also a pipe at the bottom of the tank removing 5 gal/min from the tank. Assume that the sugar is immediately and thoroughly mixed throughout the tank.

Find the amount of sugar in the tank after 10 minutes and after 2 hours.

Return

Model

- $S(t)$ = the amount of sugar in the tank in lbs.
- *Concentration* = pounds per unit volume.
  - ◇ $c(t) = \frac{S(t)}{V} \frac{\text{lbs}}{\text{gal}}$.
- Modeling is easier in terms of the total amount, $S(t)$.
- Draw a picture.

Return

Balance Law

- Rate of change = Rate in - Rate out
- Rate = volume rate × concentration
- For the problem
  - ◇ Rate in $= 5\frac{\text{gal}}{\text{min}} \times \frac{1}{2}\frac{\text{lb}}{\text{gal}} = 2.5\frac{\text{lb}}{\text{min}}$
  - ◇ Rate out $= 5\frac{\text{gal}}{\text{min}} \times \frac{S}{500}\frac{\text{lb}}{\text{gal}} = \frac{S}{100}\frac{\text{lb}}{\text{min}}$

Return

Solution

$$\frac{dS}{dt} = \text{Rate in - Rate out}$$
$$= 2.5 - \frac{S}{100}.$$

- General solution: $S(t) = 250 + Ce^{-t/100}$.
- Particular solution: $S(t) = 250(1 - e^{-t/100})$.

Return Balance law

Other possible initial conditions

- There is initially 20 lbs of sugar in the tank.
- The concentration of sugar in the tank at $t = 0$ is 1 lb/gallon.

Solution

Mixing Problem 2

A tank originally holds 500 gallons of sugar water with a concentration of $\frac{1}{10}$ lb/gal. At $t = 0$ there starts a flow of sugar water into the tank with a concentration of $\frac{1}{2}$ lbs/gal at a rate of 5 gal/min. There is also a pipe at the bottom of the tank removing 10 gal/min from the tank. Assume that the sugar is immediately and thoroughly mixed throughout the tank.

Find the amount of sugar in the tank after 10 minutes and after 2 hours.

Return

Solution

- Rate in $= 5\frac{\text{gal}}{\text{min}} \times \frac{1}{2}\frac{\text{lb}}{\text{gal}} = 2.5\frac{\text{lb}}{\text{min}}$
- Rate out $= 10\frac{\text{gal}}{\text{min}} \times \frac{S}{V}\frac{\text{lb}}{\text{gal}}$
  - ⋄ $V(t) = 500 - 5t$.
  - ⋄ Rate out $= \frac{10S}{500 - 5t}\frac{\text{lb}}{\text{min}}$

Balance law Problem Return

$$\frac{dS}{dt} = \text{Rate in - Rate out}$$
$$= 2.5 - \frac{2S}{100 - t},$$

- General solution:
$$S(t) = 2.5(100 - t) + C(100 - t)^2.$$
- Particular solution:
$$S(t) = 2.5(100 - t) - \frac{(100 - t)^2}{50}.$$

Qualitative Analysis

- Do solutions always exist?
- How many solutions are there?
  - ◇ To an initial value problem.
- Can we predict the behavior of solutions without having a formula?

Return

Example

- Initial value problem:
$$\sin(t)y' = \cos(t)y + \sin^2(t) \quad \text{with} \quad y(0) = 1.$$
- Every solution to the differential equation has the form
$$y(t) = t \sin t + C \sin t.$$
- Hence $y(0) = 0$ for every solution. The IVP with $y(0) = 1$ has *no solution.*

Questions Return

Existence of Solutions

- Put the equation $\sin(t)y' = \cos(t)y + \sin^2(t)$ into normal form

$$y' = \frac{\cos t}{\sin t}y + \sin t.$$

- The RHS is discontinuous at $t = 0$.

- If we require the RHS to be continuous there is always a solution to an initial value problem.

Example Return

Existence Theorem

Theorem: Suppose the function $f(t, y)$ is defined and continuous in the rectangle R in the ty-plane. Then given any point $(t_0, y_0) \in R$, the initial value problem

$$y' = f(t, y) \quad \text{with} \quad y(t_0) = y_0$$

has a solution $y(t)$ defined in an interval containing t_0. Furthermore the solution will be defined at least until the solution curve $t \to (t, y(t))$ leaves the rectangle R.

Example Return

What is a Theorem?

- A theorem is a logical statement.
- It contains
  - *hypotheses* (the assumptions made)
  - and *conclusions*
- The conclusions are guaranteed to be true if the hypotheses are true.

Return

Example of a "Theorem"

If it rains the sidewalks get wet.

- Hypothesis — *If it rains*
- Conclusion — *the sidewalks get wet*

Mathematics and Proof

- Theorems are proved by logical deduction.
- All of mathematics comes from a small number of very basic assumptions.
  - ◇ Called *axioms* or *postulates*.
- True of all parts of mathematics.
  - ◇ An algebraic derivation is an example of a proof.
- Definitions are not theorems.

Interval of Existence

- Example: $y' = 1 + y^2$ with $y(0) = 0$.
- RHS $f(t, y) = 1 + y^2$ is defined and continuous on the whole ty-plane. The rectangle R can be any rectangle in the plane.
- Solution $y(t) = \tan t$ "blows up" at $t = \pm\pi/2$.
- Thus the size of the rectangle on which $f(t, y)$ is continuous does not say much about the interval of existence.

Uniqueness of Solutions

- How many solutions does an initial value problem have?
- The uniqueness of solutions to an initial value problem is the mathematical equivalent of being able to predict results in science and engineering.
- We will need slightly stronger restrictions to ensure uniqueness than we needed for existence.

Questions

Uniqueness of Solutions

- Initial value problem

$$y' = y^{1/3} \quad \text{with} \quad y(0) = 0.$$

- The constant function $y_1(t) = 0$ is a solution.
- Solve by separation of variables to find that

$$y_2(t) = \begin{cases} \left(\frac{2t}{3}\right)^{3/2} & \text{, if } t > 0 \\ 0 & \text{, if } t \leq 0. \end{cases}$$

is also a solution.

Return Existence theorem

Theorem: Suppose $f(t, y)$, $\partial f/\partial y$ are continuous in the rectangle R. Let

$$M = \max_{(t,y)\in R} \left| \frac{\partial f}{\partial y}(t, y) \right|.$$

Suppose that (t_0, x_0) and (t_0, y_0) both lie in R, and

$$x' = f(t, x),\ x(t_0) = x_0 \ \&\ y' = f(t, y),\ y(t_0) = y_0.$$

Then as long as $(t, x(t))$ and $(t, y(t))$ stay in R we have

$$|x(t) - y(t)| \leq |x_0 - y_0| e^{M|t-t_0|}.$$

Existence theorem Return

Uniqueness Theorem

Theorem: Suppose the function $f(t, y)$ and its partial derivative $\partial f/\partial y$ are continuous in the rectangle R in the ty-plane. Suppose that $(t_0, x_0) \in R$. Suppose that

$$x' = f(t, x) \quad \text{and} \quad y' = f(t, y),$$

and that

$$x(t_0) = y(t_0) = x_0.$$

Then as long as $(t, x(t))$ and $(t, y(t))$ stay in R we have

$$x(t) = y(t).$$

Geometric Interpretation

- Solution curves cannot cross.
- They cannot even touch at one point.
- $y' = (y - 1)(\cos t - y)$ and $y(0) = 2$. Show $y(t) > 1$ for all t.
- $y' = y - (1 - t)^2$ and $y(0) = 0$. Show that $y(t) < 1 + t^2$ for all t.

E & U for Linear Equations

Theorem: Suppose that $a(t)$ and $g(t)$ are continuous on an interval $I = (a, b)$. Then given $t_0 \in I$ and any y_0, the initial value problem

$$y' = a(t)y + g(t) \quad \text{with} \quad y(t_0) = y_0$$

has a unique solution $y(t)$ *which exists for all* $t \in I$.

- Notice that the RHS is

$$f(t, y) = a(t)y + g(t), \quad \text{and} \quad \frac{\partial f}{\partial y} = a(t).$$

These are continuous for $t \in I$ and all y.

DFIELD5

Get a geometric look at existence and uniqueness.