Math 211

Lecture #7

Existence & Uniqueness

January 31, 2001

Qualitative Analysis

- Do solutions always exist?
- How many solutions are there?
  - ◇ To an initial value problem.
- Can we predict the behavior of solutions without having a formula?

Existence Theorem

Theorem: Suppose the function $f(t,y)$ is defined and continuous in the rectangle R in the ty-plane. Then given any point $(t_0, y_0) \in R$, the initial value problem

$$y' = f(t,y) \quad \text{with} \quad y(t_0) = y_0$$

has a solution $y(t)$ defined in an interval containing t_0. Furthermore the solution will be defined at least until the solution curve $t \to (t, y(t))$ leaves the rectangle R.

Return

Uniqueness of Solutions

- How many solutions does an initial value problem have?
- The uniqueness of solutions to an initial value problem is the mathematical equivalent of being able to predict results in science and engineering.

Example

- Initial value problem

$$y' = y^{1/3} \quad \text{with} \quad y(0) = 0.$$

- The constant function $y_1(t) = 0$ is a solution.
- Solve by separation of variables to find that

$$y_2(t) = \begin{cases} \left(\frac{2t}{3}\right)^{3/2} & \text{, if } t > 0 \\ 0 & \text{, if } t \leq 0. \end{cases}$$

is also a solution.

Uniqueness Theorem

Theorem: Suppose the function $f(t, y)$ and its partial derivative $\partial f/\partial y$ are continuous in the rectangle R in the ty-plane. Suppose that $(t_0, x_0) \in R$. Suppose that

$$x' = f(t, x) \quad \text{and} \quad y' = f(t, y),$$

and that

$$x(t_0) = y(t_0) = x_0.$$

Then as long as $(t, x(t))$ and $(t, y(t))$ stay in R we have

$$x(t) = y(t).$$

Geometric Interpretation

- Solution curves cannot cross.
- They cannot even touch at one point.
- $y' = (y-1)(\cos t - y)$ and $y(0) = 2$. Show $y(t) > 1$ for all t.
- $y' = y - (1-t)^2$ and $y(0) = 0$. Show that $y(t) < 1 + t^2$ for all t.

E & U for Linear Equations

Theorem: Suppose that $a(t)$ and $g(t)$ are continuous on an interval $I = (a, b)$. Then given $t_0 \in I$ and any y_0, the initial value problem

$$y' = a(t)y + g(t) \quad \text{with} \quad y(t_0) = y_0$$

has a unique solution $y(t)$ *which exists for all $t \in I$.*

- Notice that the RHS is

$$f(t, y) = a(t)y + g(t), \quad \text{and} \quad \frac{\partial f}{\partial y} = a(t).$$

These are continuous for $t \in I$ and all y.

DFIELD5

Get a geometric look at existence and uniqueness.

Theorem: Suppose $f(t,y)$, $\partial f/\partial y$ are continuous in the rectangle R. Let

$$M = \max_{(t,y)\in R} \left| \frac{\partial f}{\partial y}(t,y) \right|.$$

Suppose that (t_0, x_0) and (t_0, y_0) both lie in R, and

$$x' = f(t,x),\ x(t_0) = x_0 \ \&\ y' = f(t,y),\ y(t_0) = y_0.$$

Then as long as $(t, x(t))$ and $(t, y(t))$ stay in R we have

$$|x(t) - y(t)| \le |x_0 - y_0| e^{M|t-t_0|}.$$

Continuity in Initial Conditions

- Inequality: $|x(t) - y(t)| \leq |x_0 - y_0|e^{M|t-t_0|}$.
- The good news:
  - ◇ By making sure that x_0 and y_0 are very close we can make the solutions $x(t)$ and $y(t)$ close for t in an interval containing t_0.
  - ◇ Solutions are *continuous in the initial conditions.*

Return

Sensitivity with Respect to Initial Conditions

- Inequality: $|x(t) - y(t)| \leq |x_0 - y_0| e^{M|t-t_0|}$.
- The bad news:
  - As $|t - t_0|$ increases the RHS grows exponentially.
  - Over long intervals in t the solutions can get very far apart. Solutions are *sensitive to initial conditions.*

DFIELD5

Target practice with the equation
$x' = x \cos x + t \sin t$.

Try to hit $(4, -5)$, starting at $t = 0$.

Use window [0,4] X [-8,0].

Qualitative Analysis

- Ways to discover the properties of solutions without solving the equation.
- Works best with autonomous equations

$$y' = f(y)$$

- Example: $y' = \sin y$
  - ◇ Go to dfield

Properties of Autonomous Equations

- The direction field does not depend on t
- Solution curves can be translated left and right to get other solution curves.
  - If $y(t)$ is a solution, so is $y_1 = y(t + c)$ for any constant c.

Equilibrium Points & Solutions

$$y' = f(y) \qquad y' = \sin y$$

- Equilibrium point: $f(y_0) = 0$.
- Equilibrium solution: $y(t) = y_0$.
- $\sin y = 0 \quad \Leftrightarrow \quad y = k\pi, \quad k = 0, \pm 1, \ldots$
- $y' = \sin y$ has infinitely many equilibrium points and solutions:

$$y_k(t) = k\pi \quad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

Between the Equilibrium Points

$$\begin{aligned} 0 < y < \pi \; &\Rightarrow \; \sin y > 0 \\ &\Rightarrow \; y'(t) = \sin y(t) > 0 \\ &\Rightarrow \; y(t) \text{ is increasing} \end{aligned}$$

- By uniqueness, $0 < y(t) < \pi$ for all t.
- Thus $y(t) \nearrow \pi$ as $t \to \infty$
  and $y(t) \searrow 0$ as $t \to -\infty$

Between the Equilibrium Points

$$
\begin{aligned}
-\pi < y < 0 &\Rightarrow \sin y < 0 \\
&\Rightarrow y'(t) = \sin y(t) < 0 \\
&\Rightarrow y(t) \text{ is decreasing}
\end{aligned}
$$

- By uniqueness, $0 > y(t) > -\pi$ for all t.
- Thus $y(t) \searrow -\pi \quad$ as $t \to \infty$
  and $\;y(t) \nearrow 0 \quad$ as $t \to -\infty$