Math 211

Lecture #9

Population Models

February 5, 2001

Modeling Population

- Assume population changes due to births and deaths only.
- Births are roughly proportional to population

$$B = bP \quad b \text{ is the } \textit{birth rate}$$

- Deaths are roughly proportional to population

$$D = dP \quad d \text{ is the } \textit{death rate}$$

Modeling Population

- Rate of change = births − deaths

$$\frac{dP}{dt} = B - D$$
$$= bP - dP$$
$$= (b - d)P$$
$$= rP$$

- $r = b - d$ is the *reproductive rate*.

The Malthusian Model

- b and d not necessarily constants
  - ◇ Can depend on P, and perhaps also on t.
- If there exist sufficient resources in term of nutrients and space, b and d will be almost constant. Then the reproductive rate $r = b - d$ is also a constant.
- This is the *Malthusian model*.

The Malthusian Model

$$\frac{dP}{dt} = rP \quad \text{with} \quad P(0) = P_0$$

- Solution: $P(t) = P_0 e^{rt}$
  - ◇ If $r = b - d > 0$, $P(t)$ grows exponentially.
  - ◇ If $r = b - d < 0$, $P(t)$ decays exponentially.

The Malthusian Model

Under what circumstances could the Malthusian model be a good model?

- Requires unlimited resources.
- Laboratory experiments with small populations.
- Populations always outgrow the Malthusian model. This was the point that was made by Malthus.

The Logistic Model

- As the population increases individuals compete for resources — for food and for space.
- This causes the birth rate b to decrease , and the death rate d to increase.

- Birth rate $b =$ probability of an individual producing offspring in a fixed period of time.
- As $P \nearrow$, $b \searrow$ because of competition .
  - ◇ Competition results from encounters.
  - ◇ The number of encounters by one individual is roughly proportional to P.
  - ◇ $\Rightarrow$ decrease in the birth rate is $\sim P$
- $\Rightarrow \quad b = b_0 - b_1 P$

The Logistic Model

- Increase in the death rate d is $\sim P$
- $\Rightarrow \quad d = d_0 + d_1 P$
- The reproductive rate is

$$
\begin{aligned}
r &= b - d \\
&= (b_0 - b_1 P) - (d_0 + d_1 P) \\
&= (b_0 - d_0) - (b_1 + d_1) P \\
&= r_0 - r_1 P
\end{aligned}
$$

The Logistic Model

$$\frac{dP}{dt} = rP = (r_0 - r_1 P)P$$
$$= r_0 \left(1 - \frac{r_1}{r_0} P\right) P$$
$$= r_0 \left(1 - \frac{P}{K}\right) P$$

- r_0 is the *reproductive rate at small populations.*
- $K = r_0/r_1$ is the *carrying capacity.*

Analysis of the Logistic Model

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P$$

- Equation is autonomous.
- Equilibrium points are 0 & K.
- 0 is unstable, K is stable.
- $P(t) \to K$ as $t \to \infty$.
  - ◇ K is the carrying capacity.

Solution of The Logistic Model

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P \quad \text{with} \quad P(0) = P_0$$

- Solution:

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$

Estimating Parameters

- Malthusian model $P' = rP$

$$P(t) = P_0 e^{rt}$$

  - ◇ Two parameters P_0 and r.
  - ◇ Two measurements or observations needed to find the values of P_0 and r.
  - ◇ It is better to use all of the data and use least squares (linear regression).

Estimating Parameters

- Logistic model $P' = r(1 - P/K)P$

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}$$

  - ◇ Three parameters, P_0, r, and K.
  - ◇ Three measurements or observations needed to find the values of P_0, r, and K.
  - ◇ It is better to use all of the data and use least squares. (Nonlinear regression)

Modelling

- Two ways to write the rate of change of something, e.g., of a population P
  - ◇ The mathematical way is the derivative,

$$\frac{dP}{dt}.$$

  - ◇ The scientific way involves modelling, e.g.

$$r\left(1-\frac{P}{K}\right)P.$$

Modelling

- Setting the two equal gives a differential equation model

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P$$

- An equation says that two distinct mathematical expressions are equal.

Logistic Model

- Does a very good job of modeling the growth of populations under controlled circumstances.
  - In laboratory experiments.
  - In other circumstances when the situation does not change.

Logistic Model

- For human populations the model always breaks down.
  - ◇ Other factors become important, such as immigration.
  - ◇ Advance of technology.
  - ◇ Changing habits of life.