

# Math 211

Lecture #15

Systems of Linear Equations

February 19, 2001

# Example

Solve

$$3x - 4y + 5z = 3$$

$$-x + 2y - 2z = -2$$

- Find *all* solutions.
- Find a method which works for all systems, not matter how large.

Return

# Method of Solution

- Introduce the vectors

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

- and the matrix

$$C = \begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix}$$

- The system can be written as  $C\mathbf{x} = \mathbf{b}$ .

# Vectors

- A vector is a list of numbers
- 2-vectors, 3-vectors,  $n$ -vectors
  - ◇ A vector has length and direction
  - ◇ Parallel vectors are equal
- Transpose of a vector.

# Addition of Vectors

- Algebraic view of addition
- Geometric view of addition
- Addition of more than two vectors

# Multiplication by a Scalar

- Algebraic view
- Geometric view

# Linear Combinations of Vectors

- Vectors  $\mathbf{x} = (2, -3)^T$  and  $\mathbf{y} = (-1, 2)^T$ .
- Any vector of the form  $a\mathbf{x} + b\mathbf{y}$  is a *linear combination* of  $\mathbf{x}$  and  $\mathbf{y}$ .
- $3\mathbf{x} + 2\mathbf{y} = (4, -5)^T$ .
- Any 2-vector is a linear combination of  $\mathbf{x}$  and  $\mathbf{y}$ .
- Linear combinations of more than two vectors.

# Matrices

- A matrix is a rectangular array of numbers.
- Example

$$\begin{pmatrix} -1 & 0 & 2 & 6 \\ 0 & 3 & -4 & 10 \\ 3 & 3 & 2 & -5 \end{pmatrix}$$

- Size = (3,4); 3 rows & 4 columns.

# Linear Combinations and Systems

- The **example system** can be written as a vector equation

$$\begin{pmatrix} 3x - 4y + 5z \\ -x + 2y - 2z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

- or

$$x \begin{pmatrix} 3 \\ -1 \end{pmatrix} + y \begin{pmatrix} -4 \\ 2 \end{pmatrix} + z \begin{pmatrix} 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

# Coefficient Matrix

- $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ ,  $\begin{pmatrix} -4 \\ 2 \end{pmatrix}$ , and  $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$  are the column vectors in the *coefficient matrix*

$$C = \begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix}$$

- Solving the *system of equations*  $\Leftrightarrow$  finding a *linear combination* of the columns of the coefficient matrix which is equal to the RHS.

# Product of a Matrix with a Vector

- The *product* of a matrix  $A$  and a vector  $\mathbf{x}$  is the linear combination of the columns of  $A$  with the elements of  $\mathbf{x}$  as coefficients.

$$\begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ = x \begin{pmatrix} 3 \\ -1 \end{pmatrix} + y \begin{pmatrix} -4 \\ 2 \end{pmatrix} + z \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$

Return

Coefficient matrix

Linear Comb.

# Example

- Thus the system of equations becomes

$$\begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

or

$$C\mathbf{x} = \mathbf{b}$$

Product

Coefficient matrix

Linear Comb.

Solution

# Computing the Product

- From the definition.
- A faster way.
- $A\mathbf{x}$  is only defined if  $A$  is  $p \times q$  and  $\mathbf{x}$  is a column  $q$ -vector.

# Properties of the Product

- $A(a\mathbf{x}) = a(A\mathbf{x})$
- $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$
- $A(a\mathbf{x} + b\mathbf{y}) = aA\mathbf{x} + bA\mathbf{y}$
- Multiplication by a matrix is a *linear operation*.

# Product of Two Matrices

Suppose  $A$  is  $n \times p$  and  $B$  is  $p \times q$ .

Write  $B$  in terms of its column vectors

$$B = [\mathbf{b}_1 \ \mathbf{b}_2 \ \dots \ \mathbf{b}_q]$$

Define the product  $AB$  by

$$AB = [A\mathbf{b}_1 \ A\mathbf{b}_2 \ \dots \ A\mathbf{b}_q]$$

# Properties of the Product

$A$ ,  $B$ , and  $C$  matrices

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- However  $AB \neq BA$  in general

# The Identity Matrix

- In dimension 3

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- $I\mathbf{x} = \mathbf{x}$  for every 3-vector  $\mathbf{x}$ .
- $IA = A$  for every matrix  $A$  with 3 columns.
- $AI = A$  for every matrix  $A$  with 3 rows.