Math 211

Lecture #18

Properties of Solution Spaces

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Method of Solution for $A\mathbf{x} = \mathbf{b}$

- Use the augmented matrix $M = [A,\ \mathbf{b}]$.
- Eliminate as many coefficients as possible.
  - ◇ Use row operations to reduce to row echelon form.
- Write down the simplified system.
- Backsolve.
  - ◇ Assign arbitrary values to the free variables.
  - ◇ Solve for the pivot variables.

Consistent Systems

- A system is *consistent* if it has solutions.
  - The solution set is not the empty set.
- A system is consistent if and only if the simplified version (after elimination) is consistent.
- This is true if and only if the last column (after elimination) does *not* contain a pivot.

Examples

$$A = \begin{pmatrix} -3 & 6 & 0 \\ -2 & 4 & 0 \\ -1 & 0 & 2 \end{pmatrix} \quad \mathbf{b}_1 = \begin{pmatrix} 2 \\ 3 \\ -5 \end{pmatrix} \quad \mathbf{b}_2 = \begin{pmatrix} -9 \\ -6 \\ 7 \end{pmatrix}$$

Homogeneous Systems

Example

$$A = \begin{pmatrix} -5 & -4 & -2 \\ -6 & -6 & -2 \\ 30 & 27 & 11 \end{pmatrix} \Rightarrow \begin{pmatrix} -5 & -4 & -2 & 0 \\ -6 & -6 & -2 & 0 \\ 30 & 27 & 11 & 0 \end{pmatrix}$$

- During elimination the column of zeros is unchanged.
- It is unnecessary to augment a homogeneous system.

Structure of the Solution Set

Theorem: Let $\mathbf{x}_p$ be a particular solution to $A\mathbf{x}_p = \mathbf{b}$.

1. If $A\mathbf{x}_h = \mathbf{0}$ then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ also satisfies $A\mathbf{x} = \mathbf{b}$.
2. If $A\mathbf{x} = \mathbf{b}$, then there is a vector$\mathbf{x}_h$ such that $A\mathbf{x}_h = \mathbf{0}$ and $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.

- Solution set for $A\mathbf{x} = \mathbf{b}$ is known if we know one particular solution $\mathbf{x}_p$ and the solution set for the homogeneous system $A\mathbf{x}_h = \mathbf{0}$.

Square Matrices

- There are special kinds:
  - Singular and nonsingular.
  - Invertible and noninvertible.
- What do the terms mean?
- What are the relations between them?

Return

Singular and Nonsingular Matrices

The $n \times n$ matrix A is *nonsingular* if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any right hand side $\mathbf{b}$.

Proposition: The $n \times n$ matrix A is nonsingular if and only if the simplified matrix (after elimination) has only nonzero entries along the diagonal.

- In reduced row echelon form we get I.

Examples

$$A = \begin{pmatrix} -17 & -16 & -6 \\ 18 & 18 & 6 \\ 6 & 3 & 3 \end{pmatrix}$$

$$A = \begin{pmatrix} -17 & -16 & -6 \\ 18 & 18 & 6 \\ 6 & 3 & 4 \end{pmatrix}$$

Singular

Proposition: If the $n \times n$ matrix A is nonsingular then the equation $A\mathbf{x} = \mathbf{b}$ has a *unique* solution for any right hand side $\mathbf{b}$.

Proposition: The $n \times n$ matrix A is singular if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a non-zero solution.

Invertible Matrices

An $n \times n$ matrix A is *invertible* if there is an $n \times n$ matrix B such that $AB = BA = I$. The matrix B is called an *inverse* of A.

- If B_1 and B_2 are both inverses of A, then

$$B_1 = B_1(AB_2) = (B_1A)B_2 = B_2$$

- The inverse of A is denoted by A^{-1}.
- Invertible $\Rightarrow$ nonsingular .

Computing the inverse A^{-1}

- Form the matrix $[A,\ I]$.
- Do elimination until the matrix has the form $[I,\ B]$.
- Then $A^{-1} = B$.
- A matrix is invertible if and only if it is nonsingular.

Solution Set of a Homogeneous System

Our goal is to understand such sets better. In particular we want to know:

- What are the properties of these solution sets?
- Is there a convenient way to describe them?