Math 211

Lecture #19

Nullspaces and Subspaces

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Structure of the Solution Set

Theorem: Let $\mathbf{x}_p$ be a particular solution to $A\mathbf{x}_p = \mathbf{b}$.

1. If $A\mathbf{x}_h = \mathbf{0}$ then $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ also satisfies $A\mathbf{x} = \mathbf{b}$.
2. If $A\mathbf{x} = \mathbf{b}$, then there is a vector$\mathbf{x}_h$ such that $A\mathbf{x}_h = \mathbf{0}$ and $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$.

- Solution set for $A\mathbf{x} = \mathbf{b}$ is known if we know one particular solution $\mathbf{x}_p$ and the solution set for the homogeneous system $A\mathbf{x}_h = \mathbf{0}$.

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Solution Set of a Homogeneous System

Our goal is to understand such sets better. In particular we want to know:

- What are the properties of these solution sets?
- Is there a convenient way to describe them?

Nullspace of a Matrix

The *nullspace* of a matrix A is the set

$$\{\mathbf{x} \mid A\mathbf{x} = \mathbf{0}\}\,.$$

- The nullspace of A is the same as the solution set for the homogeneous system $A\mathbf{x} = \mathbf{0}$.
- The nullspace of A is denoted by $\mathrm{null}(A)$,

Properties of the Nullspace of A

Proposition: Let A be a matrix.

1. If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathrm{null}(A)$, then $\mathbf{x} + \mathbf{y}$ is in $\mathrm{null}(A)$.

2. If a is a scalar and $\mathbf{x}$ is in $\mathrm{null}(A)$, then $a\mathbf{x}$ is in $\mathrm{null}(A)$.

Subspaces of $\mathbf{R}^n$

Definition: A nonempty subset V of $\mathbf{R}^n$ that has the properties

1. if $\mathbf{x}$ and $\mathbf{y}$ are vectors in V, $\mathbf{x} + \mathbf{y}$ is in V,
2. if a is a scalar, and $\mathbf{x}$ is in V, then $a\mathbf{x}$ is in V,

is called a *subspace* of $\mathbf{R}^n$.

- The nullspace of a matrix is a subspace.

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Examples of Subspaces

- The nullspace of a matrix is a subspace.
- A line through the origin is a subspace.
  $V = \{t\mathbf{v} \,|\, t \in \mathbf{R}\}$.
- A plane through the origin is a subspace.
  $V = \{a\mathbf{v} + b\mathbf{w} \,|\, a,\, b \in \mathbf{R}\}$.
- $\{\mathbf{0}\}$ and $\mathbf{R}^n$ are subspaces of $\mathbf{R}^n$.

Linear Combinations

Proposition: Any linear combination of vectors in a subspace V is also in V.

- Subspaces of $\mathbf{R}^n$ have the same kind of linear structure as $\mathbf{R}^n$ itself.
- In particular the nullspaces of matrices have the same kind of linear structure as $\mathbf{R}^n$.

Example

$$A = \begin{pmatrix} 4 & 3 & -1 \\ -3 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$$

The nullspace of A is

$$\mathrm{null}(A) = \{a\mathbf{v} \,|\, a \in \mathbf{R}\},$$

where $\mathbf{v} = (1,\,-1,\,1)^T$.

Example

$$B = \begin{pmatrix} 4 & 3 & -1 & 6 \\ -3 & -2 & 1 & -4 \\ 1 & 2 & 1 & 4 \end{pmatrix}$$

- $\text{null}(B) = \{a\mathbf{v} + b\mathbf{w} \mid a,\ b \in \mathbf{R}\}$, where $\mathbf{v} = (1,\ -1,\ 1,\ 0)^T$ and $\mathbf{w} = (0,\ -2,\ 0,\ 1)^T$.
- $\text{null}(B)$ consists of all linear combinations of $\mathbf{v}$ and $\mathbf{w}$.

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The Span of a Set of Vectors

In every example the subspace has been the set of all linear combinations of a few vectors.

Definition: The *span* of a set of vectors is the set of all linear combinations of those vectors. The span of the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ is denoted by

$$\text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k).$$

The Span of a Set of Vectors

Proposition: If $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ are all vectors in $\mathbf{R}^n$, then $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a subspace of $\mathbf{R}^n$.

Examples

Let $\mathbf{v}_1 = (1,\, 2)^T$, $\mathbf{v}_2 = (1,\, 0)^T$, and $\mathbf{v}_3 = (2,\, 0)^T$.

- $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{R}^2$. (Proof?)
- $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_3) = \mathbf{R}^2$. (Proof?)
- $\operatorname{span}(\mathbf{v}_2, \mathbf{v}_3) = \operatorname{span}(\mathbf{v}_2)$. (Proof?)
  - $\operatorname{span}(\mathbf{v}_2, \mathbf{v}_3) = \{t\mathbf{v}_2 \,|\, t \in \mathbf{R}\}$.
  - $\mathbf{v}_2$ and $\mathbf{v}_3$ have the same direction.

Linear Independence of Two Vectors

We need a condition that will keep unneeded vectors out of a spanning list. We will work toward a general definition.

- Two vectors are *linearly dependent* if one is a scalar multiple of the other.
  - $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent.
  - $\mathbf{v}_1$ and $\mathbf{v}_2$ are *linearly independent.*

Linear Independence of Three Vectors

- Three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are *linearly dependent* if one is a linear combination of the other two.
  - ◇ Example: $\mathbf{v}_1 = (1,\, 0,\, 0)^T$, $\mathbf{v}_2 = (0,\, 1,\, 0)^T$, and $\mathbf{v}_3 = (1,\, 2,\, 0)^T$

$$\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2.$$

  - ◇ Notice that $\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$.

Linear Independence

- Three vectors are linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.
  - ◇ Non-trivial means that at least one of the coefficients is not 0.
- A set of vectors is linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.

Linear Independence

Definition: The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ are *linearly independent* if the only linear combination of them which is equal to the zero vector is the one with all of the coefficients equal to 0.

- In symbols,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0}$$

$$\Rightarrow c_1 = c_2 = \cdots = c_k = 0.$$

Basis of a Subspace

Definition: A set of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ form a *basis* of a subspace V if

1. $V = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$
2. $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , and $\mathbf{v}_k$ are linearly independent.

Examples of Bases

- The vector $\mathbf{v} = (1,\ -1,\ 1)^T$ is a basis for $\mathrm{null}(A)$.
  - $\mathrm{null}(A)$ is the subspace of $\mathbf{R}^3$ with basis $\mathbf{v}$.
- The vectors $\mathbf{v} = (1,\ -1,\ 1,\ 0)^T$ and $\mathbf{w} = (0,\ -2,\ 0,\ 1)^T$ form a basis for $\mathrm{null}(B)$.
  - $\mathrm{null}(B)$ is the subspace of $\mathbf{R}^4$ with basis $\{\mathbf{v}, \mathbf{w}\}$.

Basis of a Subspace

Proposition: Let V be a subspace of $\mathbf{R}^n$.

1. If $V \neq \{\mathbf{0}\}$, then V has a basis.
2. Every basis of V has the same number of elements.

Definition: The *dimension* of a subspace V is the number of elements in a basis of V.

Linear Independence?

How do we decide if a set of vectors is linearly independent?

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 6 \end{pmatrix}$$

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0} \Leftrightarrow$$

$$[\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]\mathbf{c} = \mathbf{0} \Leftrightarrow$$

$$\mathbf{c} \in \mathrm{null}([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]).$$

- $\mathbf{c} = (-3, 2, 1)^T \in \mathrm{null}([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3])$.

$$-3\mathbf{v}_1 + 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}.$$

- $\mathbf{v}_1, \ \mathbf{v}_2, \ \mathbf{v}_3$ are linearly dependent.

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 5 \\ 0 \\ -4 \\ 3 \end{pmatrix}$$

- $\text{null}([\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]) = \{\mathbf{0}\}$.
- $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent.

Proposition: Suppose that $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ are vectors in $\mathbf{R}^n$. Set $V = [\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k]$.

1. If $\mathrm{null}(V) = \{\mathbf{0}\}$, then $\mathbf{v}_1$, $\mathbf{v}_2$, ..., and $\mathbf{v}_k$ are linearly independent.

2. If $\mathbf{c} = (c_1, c_2, \ldots, c_k)^T$ is a nonzero vector in $\mathrm{null}(V)$, then

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0},$$

so the vectors are linearly dependent.

Method of Solution for $A\mathbf{x} = \mathbf{b}$

- Use the augmented matrix $M = [A,\ \mathbf{b}]$.
- Eliminate as many coefficients as possible.
  - Use row operations to reduce to row echelon form.
- Write down the simplified system.
- Backsolve.
  - Assign arbitrary values to the free variables.
  - Solve for the pivot variables.