Math 211

Lecture #20

Determinants

March 2, 2001

Nonsingular Matrices

A an $n \times n$ matrix

- A is *nonsingular* if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any right hand side $\mathbf{b}$.
- If A is nonsingular then $A\mathbf{x} = \mathbf{b}$ has a unique solution for any right hand side $\mathbf{b}$.
- A is singular if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a non-zero solution.
  - ⋄ $\mathrm{null}(A)$ is non-trivial $\Leftrightarrow$ A is singular.

Return

Determinants in 2D

- How do we decide if a matrix A is nonsingular?
- A is nonsingular if and only if $\texttt{rref}(A) = I$.
- $$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
  is nonsingular if and only if $ad - bc \neq 0$.
  - ⋄ if and only if $\det(A) = ad - bc \neq 0$

Return

Determinants in 3D

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\begin{aligned} \det(A) = {} & a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \\ & + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32}. \end{aligned}$$

- A is nonsingular if and only if $\det(A) \neq 0$.

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Determinants

Theorem: The $n \times n$ matrix A is nonsingular if and only if $\det(A) \neq 0$.
Corollary: If A is an $n \times n$ matrix, then $\mathrm{null}(A)$ contains a nonzero vector if and only if $\det(A) = 0$.

- The corollary contains the most important fact about determinants for ODEs.

Return

Typical Term in $\det(A)$

- Typical term $a_{13}a_{21}a_{32}$
- First subscripts are 1, 2, & 3.
  - ◇ True for all terms.
- Second subscripts are 3, 1, & 2.
  - ◇ Different order for each terms, but always a list of 1, 2, & 3.

Return

Permutations

- A *permutation* is a list of the numbers 1, 2, $\ldots$, n in any order.
  - ◇ Example $\sigma = (3, 1, 2)$.
  - ◇ Typical term is $a_{1\sigma_1}a_{2\sigma_2}a_{3\sigma_3}$
- All permutations of 3 numbers appear in the determinant of a 3×3 matrix (sometimes with minus signs).

Return

Sign of a Permutation

Transform a permutation into $(1, 2, \ldots, n)$ using interchanges.

- $(3, 1, 2) \to (1, 3, 2) \to (1, 2, 3)$
- This can be done in many ways, but for a given permutation, the number of interchanges is always odd or always even.
- A permutation is *odd* if the number is odd, and *even* if the number is even.

Return

Sign of a Permutation

Define

$$(-1)^\sigma = \begin{cases} 1 & \text{if } \sigma \text{ is even}, \\ -1 & \text{if } \sigma \text{ is odd.} \end{cases}$$

Return

Definition of Determinant

The general $n \times n$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$\det(A) = \sum_{\sigma} (-1)^{\sigma} a_{1\sigma_1} \cdot a_{2\sigma_2} \cdot \ldots \cdot a_{n\sigma_n}$$

Return Typical term Permutation Sign

Definition of Determinant

- Each summand of the definition is the product of n entries, one from each row, and one from each column.
- The definition is the sum of $n!$ terms. It does not provide an effective way to compute the determinant.
  - ◇ One exception: The determinant of an upper triagular matrix is the product of the diagonal terms.

Return

Row Operations and Determinants

If B is obtained from A by

- adding a multiple of one row to another,
  $$\det(B) = \det(A).$$
- interchanging two rows,
  $$\det(B) = -\det(A).$$
- multiplying a row by $c \neq 0$,
  $$\det(B) = c\det(A).$$

Return

Example

$$A = \begin{pmatrix} -5 & 0 & 0 \\ -25 & -4 & -14 \\ 65 & 7 & 17 \end{pmatrix}$$

$$\det(A) = -150$$

Return

Properties

- If A has two equal rows, then $\det(A) = 0$.
- $\det(A^T) = \det(A)$.
- If A has two equal columns, then $\det(A) = 0$.

Return Row operations

Column Operations and Determinants

If B is obtained from A by

- adding a multiple of one column to another,
  $$\det(B) = \det(A).$$
- interchanging two columns,
  $$\det(B) = -\det(A).$$
- multiplying a column by $c \neq 0$,
  $$\det(B) = c\det(A).$$

Return

Expansion by a Row

Definition: The ij-*minor* of an $n \times n$ matrix A is the $(n-1) \times (n-1)$ matrix A_{ij} obtained from A by deleting the i^{th} row and the j^{th} column.

With this definition we can prove that

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called *expansion by the i^{th} row.*

Expansion by a Column

We can also expand by a column.

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called *expansion by the j^{th} column.*

Example

$$A = \begin{pmatrix} -5 & -6 & 0 \\ 3 & 4 & 0 \\ -8 & -16 & 9 \end{pmatrix}$$

$$\begin{aligned} \det(A) &= 9 \cdot \det \begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix} \\ &= 9 \cdot (-2) \\ &= -18 \end{aligned}$$

Example

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -2 \\ -2 & -1 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix}$$

$$\det(A) = 1.$$

Example

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 \\ 12 & -6 & 0 & 5 \\ 32 & -15 & -3 & 13 \\ 18 & -10 & -1 & 8 \end{pmatrix}$$

$$\det(A) = -1.$$

Determinants and Bases

Proposition: A collection of n vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$,$\mathbf{v}_n$ in $\mathbf{R}^n$ is a basis for $\mathbf{R}^n$ if and only if

$$\det([\mathbf{v}_1\,\mathbf{v}_2\,\ldots\,\mathbf{v}_n]) \neq 0.$$