Math 211

Lecture #20

Systems of ODEs

March 12, 2001

Nonsingular Matrices

A an $n \times n$ matrix

- A is *nonsingular* if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any right hand side $\mathbf{b}$.
- If A is nonsingular then $A\mathbf{x} = \mathbf{b}$ has a unique solution for any right hand side $\mathbf{b}$.
- A is singular if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a non-zero solution.
  - ⋄ $\mathrm{null}(A)$ is non-trivial $\Leftrightarrow$ A is singular.

Return

Determinants

Theorem: The $n \times n$ matrix A is nonsingular if and only if $\det(A) \neq 0$.

Corollary: If A is an $n \times n$ matrix, then $\mathrm{null}(A)$ contains a nonzero vector if and only if $\det(A) = 0$.

- The corollary contains the most important fact about determinants for ODEs.

Return

Row Operations and Determinants

If B is obtained from A by

- adding a multiple of one row to another,
$$\det(B) = \det(A).$$
- interchanging two rows,
$$\det(B) = -\det(A).$$
- multiplying a row by $c \neq 0$,
$$\det(B) = c\det(A).$$

Return

Column Operations and Determinants

If B is obtained from A by

- adding a multiple of one column to another,
$$\det(B) = \det(A).$$
- interchanging two columns,
$$\det(B) = -\det(A).$$
- multiplying a column by $c \neq 0$,
$$\det(B) = c\det(A).$$

Return

Expansion by a Row

Definition: The ij-*minor* of an $n \times n$ matrix A is the $(n-1) \times (n-1)$ matrix A_{ij} obtained from A by deleting the i^{th} row and the j^{th} column.
With this definition we can prove that

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called *expansion by the i^{th} row.*

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Expansion by a Column

We can also expand by a column.

$$\det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called *expansion by the j^{th} column.*

Example

$$A = \begin{pmatrix} -5 & -6 & 0 \\ 3 & 4 & 0 \\ -8 & -16 & 9 \end{pmatrix}$$

$$\begin{aligned} \det(A) &= 9 \cdot \det \begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix} \\ &= 9 \cdot (-2) \\ &= -18 \end{aligned} .$$

Example

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -2 \\ -2 & -1 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix}$$

$$\det(A) = 1.$$

Example

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 \\ 12 & -6 & 0 & 5 \\ 32 & -15 & -3 & 13 \\ 18 & -10 & -1 & 8 \end{pmatrix}$$

$$\det(A) = -1.$$

Determinants and Bases

Proposition: A collection of n vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\ldots$,$\mathbf{v}_n$ in $\mathbf{R}^n$ is a basis for $\mathbf{R}^n$ if and only if

$$\det([\mathbf{v}_1\,\mathbf{v}_2\,\ldots\,\mathbf{v}_n]) \neq 0.$$

Systems of Differential Equations

Example: SIR model of the spread of infectious disease. Assume:

- The disease is of short duration and rarely fatal.
- The disease spreads through human contact.
- Recovered indiviuals are immune.

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SIR Model

- Three subpopulations; susceptible, $S(t)$, infecteds, $I(t)$, and recovered, $R(t)$

$$S' = -aSI$$
$$I' = aSI - bI$$
$$R' = bI.$$

- $N = S + I + R$ is constant.
- MATLAB & `pplane5`.

Return Assumptions

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General System in 2D

$$x' = f(t, x, y)$$
$$y' = g(t, x, y)$$

- Example:

$$x' = y$$
$$y' = -x$$

- Solution: $x(t) = \sin t \,\&\, y(t) = \cos t$
  - ◇ Verify by direct substitution.

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General System in Higher D

$$x_1' = f_1(t, x_1, x_2, \ldots, x_n)$$
$$x_2' = f_2(t, x_1, x_2, \ldots, x_n)$$
$$\vdots = \quad \vdots$$
$$x_n' = f_n(t, x_1, x_2, \ldots, x_n)$$

- The *dimension* of a system is the number of unknown functions = the number of equations.

Return Planar system

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Vector Notation — 2D

- In 2D set $u_1(t) = x(t)$ & $u_2(t) = y(t)$, and

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}.$$

- Then in the example

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \Leftrightarrow \mathbf{u}' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}$$

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Vector Notation — Planar System

- For the general case use vector notation and set

$$\mathbf{F}(t, \mathbf{u}) = \begin{pmatrix} f(t, u_1, u_2) \\ g(t, u_1, u_2) \end{pmatrix}.$$

- Then

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \end{aligned} \Leftrightarrow \mathbf{u}' = \mathbf{F}(t, \mathbf{u})$$

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Vector Notation — General

- In higher dimensions, set

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, \mathbf{x}) \\ f_2(t, \mathbf{x}) \\ \vdots \\ f_n(t, \mathbf{x}) \end{pmatrix}.$$

- The general system can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}).$$

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Initial Value Problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Each component of $\mathbf{x}(t_0)$ must be specified.
- Example

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \quad \text{with} \quad \begin{aligned} x(0) &= 2 \\ y(0) &= 13 \end{aligned}$$

Return

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Reduction of Higher Order Equation to a System

For any higher order equation there is a first order system which is equivalent to it, in the sense that solutions of the system lead easily to solutions of the equation, and vice versa.

- Reduces the study of higher order equations to the study of systems
- Useful for the computation of solutions of higher order equations.

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Example of Reduction

- Third-order equation: $y''' + 2yy' = 3\cos t$
- Set $x_1 = y$, $x_2 = y'$, and $x_3 = y''$.
- Then

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= 3\cos t - 2x_1x_2 \end{aligned}$$

Return

Geometric Interpretation of Solutions

- `pplane5`
- Component plot
- Parametric plot
- Phase plane
- 3-D plot
- Composite plot