

Math 211

Lecture #20

Systems of ODEs

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Nonsingular Matrices

A an $n \times n$ matrix

- A is *nonsingular* if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for any right hand side \mathbf{b} .
- If A is nonsingular then $A\mathbf{x} = \mathbf{b}$ has a unique solution for any right hand side \mathbf{b} .
- A is singular if and only if the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a non-zero solution.
 - ◇ $\text{null}(A)$ is non-trivial $\Leftrightarrow A$ is singular.

Determinants

Theorem: The $n \times n$ matrix A is nonsingular if and only if $\det(A) \neq 0$.

Corollary: If A is an $n \times n$ matrix, then $\text{null}(A)$ contains a nonzero vector if and only if $\det(A) = 0$.

- The corollary contains the most important fact about determinants for ODEs.

Row Operations and Determinants

If B is obtained from A by

- adding a multiple of one row to another,
$$\det(B) = \det(A).$$
- interchanging two rows,
$$\det(B) = -\det(A).$$
- multiplying a row by $c \neq 0$,
$$\det(B) = c \det(A).$$

Column Operations and Determinants

If B is obtained from A by

- adding a multiple of one **column** to another,
$$\det(B) = \det(A).$$
- interchanging two columns,
$$\det(B) = -\det(A).$$
- multiplying a column by $c \neq 0$,
$$\det(B) = c \det(A).$$

Expansion by a Row

Definition: The *ij*-minor of an $n \times n$ matrix A is the $(n - 1) \times (n - 1)$ matrix A_{ij} obtained from A by deleting the i^{th} row and the j^{th} column.

With this definition we can prove that

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called *expansion by the i^{th} row*.

Expansion by a Column

We can also expand by a column.

$$\det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called *expansion by the j^{th} column*.

Example

$$A = \begin{pmatrix} -5 & -6 & 0 \\ 3 & 4 & 0 \\ -8 & -16 & 9 \end{pmatrix}$$

$$\begin{aligned} \det(A) &= 9 \cdot \det \begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix} \\ &= 9 \cdot (-2) \\ &= -18 \end{aligned}$$

Example

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & -1 & -2 \\ -2 & -1 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{pmatrix}$$

$$\det(A) = 1.$$

Return

Row ops

Col ops

Row exp

Col exp

Example

$$A = \begin{pmatrix} 3 & -1 & 0 & 1 \\ 12 & -6 & 0 & 5 \\ 32 & -15 & -3 & 13 \\ 18 & -10 & -1 & 8 \end{pmatrix}$$

$$\det(A) = -1.$$

Determinants and Bases

Proposition: A collection of n vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in \mathbf{R}^n is a basis for \mathbf{R}^n if and only if

$$\det([\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n]) \neq 0.$$

Systems of Differential Equations

Example: SIR model of the spread of infectious disease. Assume:

- The disease is of short duration and rarely fatal.
- The disease spreads through human contact.
- Recovered individuals are immune.

SIR Model

- Three subpopulations; susceptible, $S(t)$, infecteds, $I(t)$, and recovered, $R(t)$

$$S' = -aSI$$

$$I' = aSI - bI$$

$$R' = bI.$$

- $N = S + I + R$ is constant.
- MATLAB & pp1ane5.

General System in 2D

$$x' = f(t, x, y)$$

$$y' = g(t, x, y)$$

- Example:

$$x' = y$$

$$y' = -x$$

- Solution: $x(t) = \sin t$ & $y(t) = \cos t$
 - ◇ Verify by direct substitution.

General System in Higher D

$$x'_1 = f_1(t, x_1, x_2, \dots, x_n)$$

$$x'_2 = f_2(t, x_1, x_2, \dots, x_n)$$

$$\vdots = \quad \quad \quad \vdots$$

$$x'_n = f_n(t, x_1, x_2, \dots, x_n)$$

- The *dimension* of a system is the number of unknown functions = the number of equations.

Vector Notation — 2D

- In 2D set $u_1(t) = x(t)$ & $u_2(t) = y(t)$, and

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}.$$

- Then in the example

$$\begin{array}{l} x' = y \\ y' = -x \end{array} \Leftrightarrow \mathbf{u}' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}$$

Vector Notation — Planar System

- For the **general case** use **vector** notation and set

$$\mathbf{F}(t, \mathbf{u}) = \begin{pmatrix} f(t, u_1, u_2) \\ g(t, u_1, u_2) \end{pmatrix}.$$

- Then

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \end{aligned} \Leftrightarrow \mathbf{u}' = \mathbf{F}(t, \mathbf{u})$$

Vector Notation — General

- In **higher** dimensions, set

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, \mathbf{x}) \\ f_2(t, \mathbf{x}) \\ \vdots \\ f_n(t, \mathbf{x}) \end{pmatrix} .$$

- The general system can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}).$$

Initial Value Problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Each **component** of $\mathbf{x}(t_0)$ must be specified.
- **Example**

$$\begin{array}{ll} x' = y & x(0) = 2 \\ y' = -x & y(0) = 13 \end{array} \quad \text{with}$$

Reduction of Higher Order Equation to a System

For any higher order equation there is a first order system which is equivalent to it, in the sense that solutions of the system lead easily to solutions of the equation, and vice versa.

- Reduces the study of higher order equations to the study of systems
- Useful for the computation of solutions of higher order equations.

Example of Reduction

- Third-order equation: $y''' + 2yy' = 3 \cos t$
- Set $x_1 = y$, $x_2 = y'$, and $x_3 = y''$.
- Then

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = 3 \cos t - 2x_1x_2$$

Geometric Interpretation of Solutions

- pp1ane5
- Component plot
- Parametric plot
- Phase plane
- 3-D plot
- Composite plot