Math 211

Lecture #22

Systems of ODEs

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Systems of Differential Equations

Example: SIR model of the spread of infectious disease. Assume:

- The disease is of short duration and rarely fatal.
- The disease spreads through human contact.
- Recovered indiviuals are immune.

SIR Model

- Three subpopulations; susceptible, $S(t)$, infecteds, $I(t)$, and recovered, $R(t)$

$$S' = -aSI$$

$$I' = aSI - bI$$

$$R' = bI.$$

- $N = S + I + R$ is constant.
- MATLAB & `pplane5`.

General System in 2D

$$x' = f(t, x, y)$$
$$y' = g(t, x, y)$$

- Example:

$$x' = y$$
$$y' = -x$$

- Solution: $x(t) = \sin t \,\&\, y(t) = \cos t$
  - ◇ Verify by direct substitution.

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General System in Higher D

$$
\begin{aligned}
x_1' &= f_1(t, x_1, x_2, \ldots, x_n) \\
x_2' &= f_2(t, x_1, x_2, \ldots, x_n) \\
\vdots &= \qquad \vdots \\
x_n' &= f_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}
$$

- The *dimension* of a system is the number of unknown functions $=$ the number of equations.

Vector Notation — 2D

- In 2D set $u_1(t) = x(t)$ & $u_2(t) = y(t)$, and

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}.$$

- Then in the example

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \Leftrightarrow \mathbf{u}' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}$$

Vector Notation — Planar System

- For the general case use vector notation and set

$$\mathbf{F}(t, \mathbf{u}) = \begin{pmatrix} f(t, u_1, u_2) \\ g(t, u_1, u_2) \end{pmatrix}.$$

- Then

$$\begin{aligned} x' &= f(t, x, y) \\ y' &= g(t, x, y) \end{aligned} \Leftrightarrow \mathbf{u}' = \mathbf{F}(t, \mathbf{u})$$

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Vector Notation — General

- In higher dimensions, set

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix} f_1(t, \mathbf{x}) \\ f_2(t, \mathbf{x}) \\ \vdots \\ f_n(t, \mathbf{x}) \end{pmatrix}.$$

- The general system can be written

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}).$$

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Initial Value Problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Each component of $\mathbf{x}(t_0)$ must be specified.
- Example

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \quad \text{with} \quad \begin{aligned} x(0) &= 2 \\ y(0) &= 13 \end{aligned}$$

Reduction of Higher Order Equation to a System

For any higher order equation there is a first order system which is equivalent to it, in the sense that solutions of the system lead easily to solutions of the equation, and vice versa.

- Reduces the study of higher order equations to the study of systems
- Useful for the computation of solutions of higher order equations.

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Example of Reduction

- Third-order equation: $y''' + 2yy' = 3\cos t$
- Set $x_1 = y$, $x_2 = y'$, and $x_3 = y''$.
- Then

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = 3\cos t - 2x_1 x_2$$

Example of Reduction

- Initial conditions in vector form

$$\mathbf{x}(t_0) = \mathbf{x}_0$$

- In component form

$$x_1(t_0) = y(t_0) = y_0,$$

$$x_2(t_0) = y'(t_0) = y_1,$$

$$x_3(t_0) = y''(t_0) = y_2.$$

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Geometric Interpretation of Solutions

- `pplane5`
- Component plot
- Parametric plot
- Phase plane
- 3-D plot
- Composite plot