# Math 211

## Lecture #23

## Linear Systems of ODEs

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# General System in 2D

$$x' = f(t, x, y)$$

$$y' = g(t, x, y)$$

- Example:

$$x' = y$$

$$y' = -x$$

# General System in Higher D

$$\begin{aligned}
x_1' &= f_1(t, x_1, x_2, \ldots, x_n) \\
x_2' &= f_2(t, x_1, x_2, \ldots, x_n) \\
\vdots &= \qquad \vdots \\
x_n' &= f_n(t, x_1, x_2, \ldots, x_n)
\end{aligned}$$

# Vector Notation — General

- set

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad \mathbf{f}(t,\mathbf{x}) = \begin{pmatrix} f_1(t,\mathbf{x}) \\ f_2(t,\mathbf{x}) \\ \vdots \\ f_n(t,\mathbf{x}) \end{pmatrix}.$$

- The general system can be written

$$\mathbf{x}' = \mathbf{f}(t,\mathbf{x}).$$

# Initial Value Problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

- Each component of $\mathbf{x}(t_0)$ must be specified.
- Example

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned} \quad \text{with} \quad \begin{aligned} x(0) &= 2 \\ y(0) &= 13 \end{aligned}$$

# Existence & Uniqueness

General System $\mathbf{x}' = \mathbf{f}(t, \mathbf{x})$

- $\mathbf{x}$ in an open set $U \subset \mathbf{R}^n$
- $t$ in an interval $I = (a, b)$
  $R = I \times U = \{(t, x) \,|\, t \in I \quad \text{and} \quad \mathbf{x} \in U\}$.

**Theorem:** Suppose that $\mathbf{f}(t, \mathbf{x})$ is continuous in $R$, and that all first partials of $\mathbf{f}$ are also continuous in $R$. Then given any $t_0 \in I$ and $\mathbf{x}_0 \in U$ there is a unique solution to the initial value problem

$$\mathbf{x}' = \mathbf{f}(t, \mathbf{x}) \quad \text{with} \quad \mathbf{x}(t_0) = \mathbf{x}_0.$$

defined on an interval containing $t_0$. The solution exists at least until the solution curve $t \rightarrow (t, \mathbf{x}(t))$ leaves $R$.

# Autonomous Systems

System of the form

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}).$$

- Look at solution curves $t \to \mathbf{x}(t) \in \mathbf{R}^n$.
- $\mathbf{R}^n$ is called *phase space*.
  - ◇ If $n = 2$ $\mathbf{R}^2$ is the phase plane.
  - ◇ If $n = 1$ $\mathbf{R}^1$ is the phase line.

# Uniqueness in Phase Space

Two solution curves in phase space for an autonomous system cannot meet at a point unless the solution curves coincide.

- If $n = 2$, two solution curves in the phase plane cannot cross, or even touch.
- *If the system is not autonomous, solution curves in the phase plane can cross.*

# Equilibrium Points & Solutions

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}).$$

- $\mathbf{x}_0$ is an *equilibrium point* if $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$.
- $\mathbf{x}(t) = \mathbf{x}_0$ is the corresponding *equilibrium solution*.
- Nullclines — where one component of $\mathbf{f}(\mathbf{x})$ vanishes.

# Example

$$x' = x^2 - y$$

$$y' = x - xy$$

- $x$-nullcline: $x^2 - y = 0$.
- $y$-nullcline: $x(1-y) = 0$.
- 3 equilibrium points: $(0,0)$, $(1,1)$, and $(-1,1)$.

Return

# Linear Systems

A system is *linear* if the unknown functions appear linearly in the right-hand sides.

- *Appear linearly* means that there are no products, powers, or higher order functions.
- Examples
  - ◇ SIR is nonlinear.
  - ◇ Previous example is linear.

Return

# Planar Linear Systems

A planar linear system is one of the form

$$x' = a(t)x + b(t)y + f(t)$$
$$y' = c(t)x + d(t)y + g(t)$$

- The coefficients can depend on $t$.

# General Linear Systems

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2 \\ \vdots &= \qquad \vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n \end{aligned}$$

- The coefficients can depend on $t$.

- Set

$$\mathbf{x} = (x_1,\, x_2,\, \ldots,\, x_n)^T$$

$$\mathbf{f}(t) = (f_1(t),\, f_2(t),\, \ldots,\, f_n(t))^T$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- The system becomes $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$.

# Existence & Uniqueness

**Theorem:** Suppose $A = A(t)$ is a matrix valued function and $\mathbf{f}(t)$ are defined and continuous in an interval $I = (\alpha, \beta)$. Then for any $t_0$ in $I$ and any $\mathbf{x}_0$ in $\mathbf{R}^n$, the initial value problem

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f} \quad \text{with} \quad \mathbf{x}(t_0) = \mathbf{x}_0$$

has a unique solution defined *for all $t$ in $I$.*

# Homogeneous Systems

$$\mathbf{x}' = A\mathbf{x}$$

**Proposition:** Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, . . . , and $\mathbf{x}_k(t)$ are solutions to the homogeneous system, and $c_1$, $c_2$, . . . , and $c_k$ are scalars. Then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_k\mathbf{x}_k(t)$$

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.