Math 211

Lecture #24

Linear Systems of ODEs

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General Linear Systems

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1 \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2 \\ \vdots &= \quad \vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n \end{aligned}$$

- The coefficients can depend on t.

- Set

$$\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$$
$$\mathbf{f}(t) = (f_1(t), f_2(t), \ldots, f_n(t))^T$$
$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- The system becomes $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$.

Homogeneous Systems

$$\mathbf{x}' = A\mathbf{x}$$

Proposition: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\ldots$, and $\mathbf{x}_k(t)$ are solutions to the homogeneous system, and c_1, c_2, $\ldots$, and c_k are scalars. Then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_k\mathbf{x}_k(t)$$

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.

Return

Very Important Example

- The system

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

has solutions

$$\mathbf{x}_1(t) = e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

  ◇ Verify by direct substitution.

Return Next

- Proposition $\Rightarrow \mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t)$ is a solution for any constants C_1 and C_2.
- Let $\mathbf{y}$ be a solution of $\mathbf{y}' = A\mathbf{y}$. Can we find C_1 and C_2 so that

$$\mathbf{y}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \quad \text{for all } t?$$

  ◇ Let's ask a simpler question.

Return Next

- Can we find C_1 and C_2 so that

$$\mathbf{y}(0) = C_1\mathbf{x}_1(0) + C_2\mathbf{x}_2(0)?$$

  ◇ We can since $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ are linearly independent.

- Uniqueness theorem ⇒

$$\mathbf{y}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \quad \text{for all } t.$$

- ⇒ Every solution to $\mathbf{x}' = A\mathbf{x}$ is a linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$.

Return Structure

Key Point in the Argument

- Need to solve the equation

$$\mathbf{y}_0 = C_1\mathbf{x}_1(0) + C_2\mathbf{x}_2(0)$$

  for any $\mathbf{y}_0 = \mathbf{y}(0)$.
- Possible if $\mathbf{x}_1(0)$ and $\mathbf{x}_2(0)$ are linearly independent.
- Uniqueness does the rest.
- We only needed $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ to be linearly independent at one point.

Return

Proposition: $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, ..., and $\mathbf{x}_k(t)$ solutions to the homogeneous system $\mathbf{x}' = A\mathbf{x}$ on the interval I.

1. If $\mathbf{x}_1(t_0)$, $\mathbf{x}_2(t_0)$, ..., and $\mathbf{x}_k(t_0)$ are linearly independent for some $t_0 \in I$, then they are linearly independent for all $t \in I$.
2. If $\mathbf{x}_1(t_0)$, $\mathbf{x}_2(t_0)$, ..., and $\mathbf{x}_k(t_0)$ are linearly dependent for some $t_0 \in I$, then they are linearly dependent for all $t \in I$.

Return Example

Linear Independence

Definition: A set of k solutions to the linear system $\mathbf{x}' = A\mathbf{x}$ is *linearly independent* if they are linearly independent at one value of t.

- Proposition $\Rightarrow$ the solutions are linearly independent for all values of t.

Structure of the Solution Space

Theorem: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\ldots$, and $\mathbf{x}_n(t)$ are linearly independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ on the interval I. Then every solution is a linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\ldots$, and $\mathbf{x}_n(t)$.

- That is, if $\mathbf{x}(t)$ is a solution, then there are constants C_1, C_2, $\ldots$, and C_n such that

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) + \cdots + C_n\mathbf{x}_n(t).$$

Solution Strategy

- The obvious strategy for completely solving the system is to look for n linearly independent solutions. solutions.

Definition: A set of n linear independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ is called a *fundamental set of solutions.*

- We will look for fundamental sets of solutions.

Example 1: $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\mathbf{x}_1(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

are a fundamental set of solutions.

Return Linear independence

Example 2: $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$$

$$\mathbf{x}_1(t) = e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

are a fundamental set of solutions.

Linear independence

Linear Systems with Constant Coefficients

- Homogeneous equations first.
- These are equations which we can solve exactly.
- We will start with the easiest case to motivate what we do.

Dimension $= 1$

- One equation: $x' = ax$
  - ⋄ a is a constant.
- Solution: $x(t) = Ce^{at}$
- Solutions are exponentials. Can we find exponential solutions to a system of equations?

Return

Exponential Solutions to $\mathbf{x}' = A\mathbf{x}$

- Look for solution of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, $\mathbf{v}$ a vector with constant entries.

$$\mathbf{x}' = \lambda e^{\lambda t}\mathbf{v}$$

$$A\mathbf{x} = e^{\lambda t}A\mathbf{v}$$

$$\mathbf{x}' = A\mathbf{x} \quad \Leftrightarrow \quad A\mathbf{v} = \lambda\mathbf{v}$$

- If $A\mathbf{v} = \lambda\mathbf{v}$ then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Can we find λ and $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$?

Return Dimension $= 1$

Eigenvalues & Eigenvectors

Definition: λ is an *eigenvalue* of A if there is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

If λ is an eigenvalue of A, then any vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ is called an *eigenvector associated with* λ.

- λ an eigenvalue of A, $\mathbf{v}$ an associated eigenvector $\Rightarrow$ $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$.

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Finding Eigenvalues

λ is an eigenvalue of A

$\Leftrightarrow$ there is a vector $\mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

$\Leftrightarrow$ $\mathbf{v} \neq \mathbf{0}$ and

$$\begin{aligned} \mathbf{0} &= A\mathbf{v} - \lambda\mathbf{v} \\ &= A\mathbf{v} - \lambda I\mathbf{v} \\ &= (A - \lambda I)\mathbf{v} \end{aligned}$$

$\Leftrightarrow$ $A - \lambda I$ has a nontrivial nullspace.

$\Leftrightarrow$ $\det(A - \lambda I) = 0$.

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Example

$$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} -4-\lambda & 2 \\ -3 & 1-\lambda \end{pmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (-4-\lambda)(1-\lambda) + 6 \\ &= \lambda^2 + 3\lambda + 2 \\ &= (\lambda+1)(\lambda+2) \end{aligned}$$

- A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$.

Return

$\det(A - \lambda I)$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{pmatrix}$$

Return Finding eigenvalues

- If A is an $n \times n$ matrix $p(\lambda)$ is a polynomial of degree n.

Definition: The *characteristic polynomial* of the $n \times n$ matrix A is

$$p(\lambda) = \det(A - \lambda I).$$

The *characteristic equation* is $p(\lambda) = 0$.

- The eigenvalues of A are the roots of the characteristic equation.

- Usually $p(\lambda) = \det(A - \lambda I) = 0$ has n roots. Usually A has n eigenvalues.
- Each eigenvalue has an associated eigenvector.
- Each eigenvector leads to a solution .
- Expect n different solutions. Are they linearly independent?

Finding Eigenvectors

- $\mathbf{v}$ is an eigenvector associated with λ if

$$\begin{aligned} &A\mathbf{v} = \lambda\mathbf{v} \\ \Leftrightarrow\ &(A - \lambda I)\mathbf{v} = \mathbf{0} \\ \Leftrightarrow\ &\mathbf{v} \in \text{null}(A - \lambda I) \end{aligned}$$

- The set of all eigenvectors associated to the eigenvalue λ is equal to the nullspace of $A - \lambda I$. It is a subspace of $\mathbf{R}^n$ called the *eigenspace* of λ.

Example

$$A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

- A has eigenvalues

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = -2.$$

- Look for associated eigenvectors.
- Look for solutions to $\mathbf{x}' = A\mathbf{x}$.

- $\lambda_1 = -1$

$$A - \lambda_1 I = \begin{pmatrix} -4+1 & 2 \\ -3 & 1+1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix}$$

$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector

$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 = e^{-t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is a solution .

- $\lambda_2 = -2$

$$A - \lambda_2 I = \begin{pmatrix} -4+2 & 2 \\ -3 & 1+2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix}$$

$\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector

$\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2 = e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a solution .

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

- The system has solutions

$$\mathbf{x}_1(t) = e^{-t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

- $x_1(0) = \mathbf{v}_1$ and $\mathbf{x}_2(0) = \mathbf{v}_2$ are linearly independent .
- $\mathbf{x}_1$ and $\mathbf{x}_2$ form a fundamental set of solutions.

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix}$$

- The general solution is the set of all linear combinations:

$$\begin{aligned} \mathbf{x}(t) &= C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) \\ &= C_1 e^{-t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2C_1 e^{-t} + C_2 e^{-2t} \\ 3C_1 e^{-t} + C_2 e^{-2t} \end{pmatrix} \end{aligned}$$

Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A
  - ◇ the roots of $\det(A - \lambda I) = 0$
- For each eigenvalue λ find the eigenspace
  - ◇ $= \text{null}(A - \lambda I)$
- If λ is an eigenvalue and $\mathbf{v}$ is an associated eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Show that n of these are linearly independent.