Math 211

Lecture #25

Exponential Solutions

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General Linear Systems

$$\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1 \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2 \\
\vdots &= \qquad \vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n
\end{aligned}$$

- The coefficients can depend on t.

Return

- Set

$$\begin{aligned}
\mathbf{x} &= (x_1,\, x_2,\, \dots,\, x_n)^T \\
\mathbf{f}(t) &= (f_1(t),\, f_2(t),\, \dots,\, f_n(t))^T \\
A &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}
\end{aligned}$$

- The system becomes $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$.

Return Previous

Homogeneous Systems

$$\mathbf{x}' = A\mathbf{x}$$

Proposition: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\dots$, and $\mathbf{x}_k(t)$ are solutions to the homogeneous system, and c_1, c_2, $\dots$, and c_k are scalars. Then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_k\mathbf{x}_k(t)$$

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.

Return

Linear Independence

Definition: A set of k solutions to the linear system $\mathbf{x}' = A\mathbf{x}$ is *linearly independent* if they are linearly independent at one value of t.

- Proposition $\Rightarrow$ the solutions are linearly independent for all values of t.

Return

Structure of the Solution Space

Theorem: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\dots$, and $\mathbf{x}_n(t)$ are linearly independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ on the interval I. Then every solution is a linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, $\dots$, and $\mathbf{x}_n(t)$.

- That is, if $\mathbf{x}(t)$ is a solution, then there are constants C_1, C_2, $\dots$, and C_n such that

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) + \cdots + C_n\mathbf{x}_n(t).$$

Return

Solution Strategy

- The obvious strategy for completely solving the system is to look for n linearly independent solutions. solutions.
  Definition: A set of n linear independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ is called a *fundamental set of solutions.*
- We will look for fundamental sets of solutions.

Return

Linear Systems with Constant Coefficients

- Homogeneous equations first.
- These are equations which we can solve exactly.
- We will start with the easiest case to motivate what we do.

Eigenvalues & Eigenvectors

Definition: λ is an *eigenvalue* of A if there is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$.

If λ is an eigenvalue of A, then any vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ is called an *eigenvector associated with* λ.

- λ an eigenvalue of A, $\mathbf{v}$ an associated eigenvector $\Rightarrow$ $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$.

Return

Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A
  - ◇ the roots of $p(\lambda) = \det(A - \lambda I) = 0$
- For each eigenvalue λ find the eigenspace
  - ◇ $= \mathrm{null}(A - \lambda I)$
- If λ is an eigenvalue and $\mathbf{v}$ is an associated eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Show that n of these are linearly independent.

Return

Cases

- Distinct real eigenvalues.
  - ◇ In this case the method works as described.
- Complex eigenvalues.
  - ◇ The method yields complex solutions.
- Repeated eigenvalues.
  - ◇ The method does not always give enough solutions.

Return

Planar System $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

In nonvector form

$$x_1' = a_{11}x_1 + a_{12}x_2$$
$$x_2' = a_{21}x_1 + a_{22}x_2$$

Return Procedure

Characteristic Polynomial

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} \\ &= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} \\ &= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) \end{aligned}$$

Return Procedure

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- Set $D = \det(A) = a_{11}a_{22} - a_{12}a_{21}$
- The *trace* of A is $\mathrm{tr}(A) = a_{11} + a_{22}$.
  Set $T = \mathrm{tr}(A)$.
- Then
  $$p(\lambda) = \lambda^2 - T\lambda + D.$$
- The eigenvalues of A are the roots of p.

Return Procedure

Eigenvalues of A

- Roots of $p(\lambda) = \lambda^2 - T\lambda + D = 0$.
  $$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$
- Three cases:
  - ◇ 2 distinct real roots if $T^2 - 4D > 0$
  - ◇ 2 complex conjugate roots if $T^2 - 4D < 0$
  - ◇ Double real root if $T^2 - 4D = 0$

Return

Eigenvectors are LI

The problem of determining that solutions are linearly independent is eased by the following.

Proposition: $\lambda_1 \neq \lambda_2$ eigenvalues of A.

$\mathbf{v}_1 \neq 0$ and $\mathbf{v}_2 \neq 0$ eigenvectors associated with λ_1 and λ_2, resp. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Return

Two Distinct Real Eigenvalues

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2}, \quad \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

- $T^2 - 4D > 0$ so $\lambda_1 < \lambda_2$.
- Associated nonzero eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$.
- Solutions $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2$.
- $\lambda_1 \neq \lambda_2 \quad \Rightarrow \quad \mathbf{x}_1(0) = \mathbf{v}_1$ and $\mathbf{x}_2(0) = \mathbf{v}_2$ are linearly independent .

Return Cases

Two Distinct Real Eigenvalues

- If A is a 2×2 matrix with
  - ◇ two distinct real eigenvalues $\lambda_1 \neq \lambda_2$, and
  - ◇ associated nonzero eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$
- Then $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2$ are a fundamental set of solutions.
  - ◇ The general solution is

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2.$$

Return

Example

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -6 & -8 \\ 4 & 6 \end{pmatrix}$$

- Characteristic polynomial:

$$p(\lambda) = \lambda^2 - 4.$$

- Eigenvalues:

$$\lambda_1 = -2 \quad \text{and} \quad \lambda_2 = 2.$$

$\lambda_1 = -2$

- $A - \lambda I = \begin{pmatrix} -4 & -8 \\ 4 & 8 \end{pmatrix}.$
- Eigenvector $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
- Solution $\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 = e^{-2t}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\lambda_2 = 2$

- $A - \lambda I = \begin{pmatrix} -8 & -8 \\ 4 & 4 \end{pmatrix}.$
- Eigenvector $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
- Solution $\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2 = e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$
- $\mathbf{x}_1$ and $\mathbf{x}_2$ are a fundamental set of solutions.
- The general solution is

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t).$$

Initial Value Problem

Solve

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -6 & -8 \\ 4 & 6 \end{pmatrix},$$

with the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

Return Fundamental set of solutions λ_1

The solution is

$$\begin{aligned}\mathbf{x}(t) &= -5\mathbf{x}_1(t) + 9\mathbf{x}_2(t) \\ &= -5e^{-2t}\begin{pmatrix} -2 \\ 1 \end{pmatrix} + 9e^{2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 10e^{-2t} - 9e^{2t} \\ -5e^{-2t} + 9e^{2t} \end{pmatrix}.\end{aligned}$$

Complex Eigenvalues

- If the discriminant $T^2 - 4D < 0$ we have complex eigenvalues

$$\lambda = \frac{T + i\sqrt{4D - T^2}}{2}, \quad \overline{\lambda} = \frac{T - i\sqrt{4D - T^2}}{2}$$

- Example: $\begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$. $p(\lambda) = \lambda^2 - 2\lambda + 5$.
  - ◇ Eigenvalues: $\lambda = 1 + 2i$ and $\overline{\lambda} = 1 - 2i$
  - ◇ Eigenvector: $\mathbf{w} = \begin{pmatrix} 3 - i \\ 1 \end{pmatrix}$

Return

Complex Numbers

A *complex number* is one of the form $z = x + iy$, where x and y are real numbers.

- $i^2 = -1$.
- x is the *real part* of z; $x = \mathrm{Re} z$.
- y is the *imaginary part* of z; $y = \mathrm{Im} z$.
  - ◇ The imaginary part is a *real* number.
- Addition and multiplication.

Return

Complex Conjugate

Definition: The *conjugate* of $z = x + iy$ is $\overline{z} = x - iy$.

- $z = \overline{z} \Leftrightarrow z$ is a real number.
- $x = \mathrm{Re} z = \dfrac{z + \overline{z}}{2}$; $y = \mathrm{Im} z = \dfrac{z - \overline{z}}{2i}$
- $\overline{z + w} = \overline{z} + \overline{w}$; $\overline{z - w} = \overline{z} - \overline{w}$
- $\overline{zw} = \overline{z} \cdot \overline{w}$; $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$

Return

Absolute Value

Definition: The *absolute value* of $z = x + iy$ is the real number $|z| = \sqrt{x^2 + y^2}$.

- $z \cdot \overline{z} = |z|^2 = x^2 + y^2$.
- $|zw| = |z||w|$
- $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$

Return

Quotients

- The reciprocal of $z = x + iy$

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{\overline{z}}{z\overline{z}} = \frac{\overline{z}}{|z|^2}.$$

$$\frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

- The quotient z/w

$$\frac{z}{w} = z \cdot \frac{1}{w} = z \cdot \frac{\overline{w}}{|w|^2} = \frac{z\overline{w}}{|w|^2}$$

Return

Geometric Representation

- Complex plane.
  - ◇ $z = x + iy \;\leftrightarrow\; (x, y)$.
- Addition
  - ◇ Parallelogram rule
- Conjugate

Return

Polar Representation

- $z = x + iy = r[\cos\theta + i\sin\theta]$.
  - ◇ θ is the *argument* of z: $\quad \tan\theta = y/x$.
  - ◇ $r = |z|$.
- *Euler's formula:* $e^{i\theta} = \cos\theta + i\sin\theta$.
  - ◇ $z = |z|e^{i\theta}$.
  - ◇ $\overline{z} = |z|e^{-i\theta}$.

Return

Multiplication

- Two complex numbers

$$z = |z|e^{i\theta} \quad \text{and} \quad w = |w|e^{i\phi}$$

- The product is

$$zw = |z|e^{i\theta} \cdot |w|e^{i\phi} = |z||w|e^{i(\theta+\phi)}.$$

- $|zw| = |z||w|$.
- The argument of zw is the sum of the arguments of z and w.

Return

Complex Exponential

Definition: For $z = x + iy$ we define

$$e^z = e^{x+iy} = e^x \cdot \, e^{iy} \, = e^x[\cos y + i \sin y].$$

Properties:

- $e^{z+w} = e^z \cdot e^w; \quad e^{z-w} = e^z \cdot e^{-w} = e^z/e^w$
- $\overline{e^z} = e^{\overline{z}}$
- $|e^z| = e^x = e^{\mathrm{Re}\, z}$
- If λ is a complex number, then $\dfrac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$

Return