

Math 211

Lecture #25

Exponential Solutions

March 21, 2001

General Linear Systems

$$x'_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1$$

$$x'_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2$$

$$\vdots = \quad \vdots$$

$$x'_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n$$

- The coefficients can depend on t .

- Set

$$\mathbf{x} = (x_1, x_2, \dots, x_n)^T$$

$$\mathbf{f}(t) = (f_1(t), f_2(t), \dots, f_n(t))^T$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

- The system becomes $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$.

Homogeneous Systems

$$\mathbf{x}' = A\mathbf{x}$$

Proposition: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, \dots , and $\mathbf{x}_k(t)$ are solutions to the homogeneous system, and c_1 , c_2 , \dots , and c_k are scalars. Then

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \dots + c_k\mathbf{x}_k(t)$$

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.

Linear Independence

Definition: A set of k solutions to the linear system $\mathbf{x}' = A\mathbf{x}$ is *linearly independent* if they are linearly independent at one value of t .

- Proposition \Rightarrow the solutions are linearly independent for all values of t .

Structure of the Solution Space

Theorem: Suppose that $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, \dots , and $\mathbf{x}_n(t)$ are **linearly independent** solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ on the interval I . Then every solution is a linear combination of $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, \dots , and $\mathbf{x}_n(t)$.

- That is, if $\mathbf{x}(t)$ is a solution, then there are constants C_1 , C_2 , \dots , and C_n such that

$$\mathbf{x}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) + \cdots + C_n\mathbf{x}_n(t).$$

Solution Strategy

- The obvious **strategy** for completely solving the system is to look for n linearly independent solutions. solutions.

Definition: A set of n linear independent solutions to the $n \times n$ homogeneous system $\mathbf{x}' = A\mathbf{x}$ is called a *fundamental set of solutions*.

- We will look for fundamental sets of solutions.

Linear Systems with Constant Coefficients

- Homogeneous equations first.
- These are equations which we can solve exactly.
- We will start with the easiest case to motivate what we do.

Eigenvalues & Eigenvectors

Definition: λ is an *eigenvalue* of A if there is a nonzero vector \mathbf{v} such that $A\mathbf{v} = \lambda\mathbf{v}$.

If λ is an eigenvalue of A , then any vector \mathbf{v} such that $A\mathbf{v} = \lambda\mathbf{v}$ is called an *eigenvector associated with λ* .

- λ an eigenvalue of A , \mathbf{v} an associated eigenvector $\Rightarrow \mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution to $\mathbf{x}' = A\mathbf{x}$.

Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A
 - ◇ the roots of $p(\lambda) = \det(A - \lambda I) = 0$
- For each eigenvalue λ find the eigenspace
 - ◇ $= \text{null}(A - \lambda I)$
- If λ is an eigenvalue and \mathbf{v} is an associated eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Show that n of these are linearly independent.

Cases

- Distinct real eigenvalues.
 - ◇ In this case the method works as described.
- Complex eigenvalues.
 - ◇ The method yields complex solutions.
- Repeated eigenvalues.
 - ◇ The method does not always give enough solutions.

Planar System $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

In nonvector form

$$x_1' = a_{11}x_1 + a_{12}x_2$$

$$x_2' = a_{21}x_1 + a_{22}x_2$$

Characteristic Polynomial

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$p(\lambda) = \det(A - \lambda I)$$

$$= \det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix}$$

$$= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- Set $D = \det(A) = a_{11}a_{22} - a_{12}a_{21}$
- The *trace* of A is $\text{tr}(A) = a_{11} + a_{22}$.
Set $T = \text{tr}(A)$.

- Then

$$p(\lambda) = \lambda^2 - T\lambda + D.$$

- The eigenvalues of A are the roots of p .

Eigenvalues of A

- Roots of $p(\lambda) = \lambda^2 - T\lambda + D = 0$.

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- Three cases:
 - ◇ 2 distinct real roots if $T^2 - 4D > 0$
 - ◇ 2 complex conjugate roots if $T^2 - 4D < 0$
 - ◇ Double real root if $T^2 - 4D = 0$

Eigenvectors are LI

The problem of determining that solutions are linearly independent is eased by the following.

Proposition: $\lambda_1 \neq \lambda_2$ eigenvalues of A .

$\mathbf{v}_1 \neq 0$ and $\mathbf{v}_2 \neq 0$ eigenvectors associated with λ_1 and λ_2 , resp. Then \mathbf{v}_1 and \mathbf{v}_2 are linearly independent.

Two Distinct Real Eigenvalues

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2}, \quad \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

- $T^2 - 4D > 0$ so $\lambda_1 < \lambda_2$.
- Associated nonzero eigenvectors \mathbf{v}_1 and \mathbf{v}_2 .
- Solutions $\mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda_2 t} \mathbf{v}_2$.
- $\lambda_1 \neq \lambda_2 \Rightarrow \mathbf{x}_1(0) = \mathbf{v}_1$ and $\mathbf{x}_2(0) = \mathbf{v}_2$ are linearly **independent**.

Two Distinct Real Eigenvalues

- If A is a 2×2 matrix with
 - ◇ two distinct real eigenvalues $\lambda_1 \neq \lambda_2$, and
 - ◇ associated nonzero eigenvectors \mathbf{v}_1 and \mathbf{v}_2
- Then $\mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda_2 t} \mathbf{v}_2$ are a fundamental set of solutions.
 - ◇ The general solution is

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2.$$

Example

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -6 & -8 \\ 4 & 6 \end{pmatrix}$$

- Characteristic polynomial:

$$p(\lambda) = \lambda^2 - 4.$$

- Eigenvalues:

$$\lambda_1 = -2 \quad \text{and} \quad \lambda_2 = 2.$$

$$\lambda_1 = -2$$

- $A - \lambda I = \begin{pmatrix} -4 & -8 \\ 4 & 8 \end{pmatrix}$.
- Eigenvector $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$
- Solution $\mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{v}_1 = e^{-2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$\lambda_2 = 2$$

- $A - \lambda I = \begin{pmatrix} -8 & -8 \\ 4 & 4 \end{pmatrix}$.
- Eigenvector $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
- Solution $\mathbf{x}_2(t) = e^{\lambda_2 t} \mathbf{v}_2 = e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$
- \mathbf{x}_1 and \mathbf{x}_2 are a **fundamental set** of solutions.
- The general solution is

$$\mathbf{x}(t) = C_1 \mathbf{x}_1(t) + C_2 \mathbf{x}_2(t).$$

Initial Value Problem

Solve

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -6 & -8 \\ 4 & 6 \end{pmatrix},$$

with the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

The solution is

$$\begin{aligned}\mathbf{x}(t) &= -5\mathbf{x}_1(t) + 9\mathbf{x}_2(t) \\ &= -5e^{-2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + 9e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 10e^{-2t} - 9e^{2t} \\ -5e^{-2t} + 9e^{2t} \end{pmatrix}.\end{aligned}$$

Complex Eigenvalues

- If the **discriminant** $T^2 - 4D < 0$ we have complex **eigenvalues**

$$\lambda = \frac{T + i\sqrt{4D - T^2}}{2}, \quad \bar{\lambda} = \frac{T - i\sqrt{4D - T^2}}{2}$$

- Example: $\begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$. $p(\lambda) = \lambda^2 - 2\lambda + 5$.

◇ Eigenvalues: $\lambda = 1 + 2i$ and $\bar{\lambda} = 1 - 2i$

◇ Eigenvector: $\mathbf{w} = \begin{pmatrix} 3 - i \\ 1 \end{pmatrix}$

Complex Numbers

A *complex number* is one of the form $z = x + iy$, where x and y are real numbers.

- $i^2 = -1$.
- x is the *real part* of z ; $x = \operatorname{Re}z$.
- y is the *imaginary part* of z ; $y = \operatorname{Im}z$.
 - ◇ The imaginary part is a *real* number.
- Addition and multiplication.

Complex Conjugate

Definition: The *conjugate* of $z = x + iy$ is $\bar{z} = x - iy$.

- $z = \bar{z} \Leftrightarrow z$ is a real number.
- $x = \operatorname{Re}z = \frac{z + \bar{z}}{2}$; $y = \operatorname{Im}z = \frac{z - \bar{z}}{2i}$
- $\overline{z + w} = \bar{z} + \bar{w}$; $\overline{z - w} = \bar{z} - \bar{w}$
- $\overline{zw} = \bar{z} \cdot \bar{w}$; $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

Absolute Value

Definition: The *absolute value* of $z = x + iy$ is the real number $|z| = \sqrt{x^2 + y^2}$.

- $z \cdot \bar{z} = |z|^2 = x^2 + y^2$.
- $|zw| = |z||w|$
- $\left| \frac{z}{w} \right| = \frac{|z|}{|w|}$

Quotients

- The reciprocal of $z = x + iy$

$$\frac{1}{z} = \frac{1}{z} \cdot \frac{\bar{z}}{\bar{z}} = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}.$$

$$\frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$$

- The quotient z/w

$$\frac{z}{w} = z \cdot \frac{1}{w} = z \cdot \frac{\bar{w}}{|w|^2} = \frac{z\bar{w}}{|w|^2}$$

Geometric Representation

- Complex plane.
 - ◇ $z = x + iy \leftrightarrow (x, y)$.
- Addition
 - ◇ Parallelogram rule
- Conjugate

Polar Representation

- $z = x + iy = r[\cos \theta + i \sin \theta]$.
 - ◇ θ is the *argument* of z : $\tan \theta = y/x$.
 - ◇ $r = |z|$.
- *Euler's formula*: $e^{i\theta} = \cos \theta + i \sin \theta$.
 - ◇ $z = |z|e^{i\theta}$.
 - ◇ $\bar{z} = |z|e^{-i\theta}$.

Multiplication

- Two complex numbers

$$z = |z|e^{i\theta} \quad \text{and} \quad w = |w|e^{i\phi}$$

- The product is

$$zw = |z|e^{i\theta} \cdot |w|e^{i\phi} = |z||w|e^{i(\theta+\phi)}.$$

- $|zw| = |z||w|.$
- The argument of zw is the sum of the arguments of z and w .

Complex Exponential

Definition: For $z = x + iy$ we define

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x [\cos y + i \sin y].$$

Properties:

- $e^{z+w} = e^z \cdot e^w$; $e^{z-w} = e^z \cdot e^{-w} = e^z / e^w$
- $\overline{e^z} = e^{\bar{z}}$
- $|e^z| = e^x = e^{\operatorname{Re}z}$
- If λ is a complex number, then $\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$