

Math 211

Lecture #26

Solutions of a Planar System

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Polar Representation

- $z = x + iy = r[\cos \theta + i \sin \theta]$.
 - ◊ θ is the *argument* of z : $\tan \theta = y/x$.
 - ◊ $r = |z|$.
- *Euler's formula*: $e^{i\theta} = \cos \theta + i \sin \theta$.
 - ◊ $z = |z|e^{i\theta}$.
 - ◊ $\bar{z} = |z|e^{-i\theta}$.

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Multiplication

- Two complex numbers

$$z = |z|e^{i\theta} \quad \text{and} \quad w = |w|e^{i\phi}$$
- The product is

$$zw = |z|e^{i\theta} \cdot |w|e^{i\phi} = |z||w|e^{i(\theta+\phi)}.$$
- $|zw| = |z||w|$.
- The argument of zw is the sum of the arguments of z and w .

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Complex Exponential

Definition: For $z = x + iy$ we define

$$e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x [\cos y + i \sin y].$$

Properties:

- $e^{z+w} = e^z \cdot e^w$; $e^{z-w} = e^z \cdot e^{-w} = e^z / e^w$
- $\overline{e^z} = e^{\bar{z}}$
- $|e^z| = e^x = e^{\operatorname{Re}z}$
- If λ is a complex number, then $\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$

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Complex Matrices

Matrices (or vectors) with complex entries inherit many of the properties of complex numbers.

- $M = A + iB$ where $A = \operatorname{Re}M$ and $B = \operatorname{Im}M$ are real matrices.
- $\overline{\overline{M}} = M$; $M = \overline{M} \Leftrightarrow M$ is real.
- $\operatorname{Re}M = \frac{1}{2}(M + \overline{M})$; $\operatorname{Im}M = \frac{1}{2i}(M - \overline{M})$
- $\overline{M + N} = \overline{M} + \overline{N}$
- $\overline{Mz} = \overline{M}\overline{z}$

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Procedure to Solve $\mathbf{x}' = A\mathbf{x}$

- Find the eigenvalues of A
 - ◊ the roots of $p(\lambda) = \det(A - \lambda I) = 0$
- For each eigenvalue λ find the eigenspace
 - ◊ $= \operatorname{null}(A - \lambda I)$
- If λ is an eigenvalue and \mathbf{v} is an associated eigenvector, $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Show that n of these are linearly independent.

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Cases

- Distinct real eigenvalues.
 - ◊ In this case the method works as described.
- Complex eigenvalues.
 - ◊ The method yields complex solutions.
- Repeated eigenvalues.
 - ◊ The method does not always give enough solutions.

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Complex Eigenpairs

A a real matrix

- λ a complex eigenvalue with associated eigenvector \mathbf{w} , so $A\mathbf{w} = \lambda\mathbf{w}$.

$$\overline{A\mathbf{w}} = \overline{\lambda\mathbf{w}} = A\overline{\mathbf{w}}$$

$$\overline{\lambda\mathbf{w}} = \overline{\lambda}\overline{\mathbf{w}}$$

- $A\mathbf{w} = \lambda\mathbf{w} \Rightarrow \overline{A\mathbf{w}} = \overline{\lambda\mathbf{w}} \Rightarrow A\overline{\mathbf{w}} = \overline{\lambda}\overline{\mathbf{w}}$
- $\Rightarrow \overline{\lambda}$ is an eigenvalue of A with associated eigenvector $\overline{\mathbf{w}}$

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Complex Matrices

- Thus complex eigenvalues come in conjugate pairs λ and $\overline{\lambda}$.
- The associated eigenvectors also come in conjugate pairs \mathbf{w} and $\overline{\mathbf{w}}$.
- $\lambda \neq \overline{\lambda} \Rightarrow \mathbf{w}$ and $\overline{\mathbf{w}}$ are linearly independent.

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Complex Matrices

- Complex exponential solutions

$$\mathbf{z}(t) = e^{\lambda t} \mathbf{w} \quad \text{and} \quad \bar{\mathbf{z}}(t) = e^{\bar{\lambda} t} \bar{\mathbf{w}}.$$

- \mathbf{z} and $\bar{\mathbf{z}}$ are linearly independent complex valued solutions to $\mathbf{x}' = A\mathbf{x}$.

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Complex Matrices

Complex eigenpairs

$$\mathbf{z}(t) = \mathbf{x}(t) + i\mathbf{y}(t) \quad \& \quad \bar{\mathbf{z}}(t) = \mathbf{x}(t) - i\mathbf{y}(t)$$

$$\mathbf{x}(t) = \operatorname{Re}(\mathbf{z}(t)) = \frac{\mathbf{z}(t) + \bar{\mathbf{z}}(t)}{2}$$

$$\mathbf{y}(t) = \operatorname{Im}(\mathbf{z}(t)) = \frac{\mathbf{z}(t) - \bar{\mathbf{z}}(t)}{2i}$$

- $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are real valued solutions.
- $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are linearly independent.

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Complex Matrices

Complex eigenpairs

Complex solutions

Planar System $\mathbf{x}' = A\mathbf{x}$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{and} \quad \mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

- Characteristic polynomial:

$$p(\lambda) = \lambda^2 - T\lambda + D.$$

$$\diamond T = \operatorname{tr} A = a_{11} + a_{22}; \quad D = \det A$$

$$\diamond \text{The eigenvalues of } A \text{ are the roots of } p.$$

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Eigenvalues of A

- Roots of $p(\lambda) = \lambda^2 - T\lambda + D = 0$.

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- Three cases:
 - ◊ 2 distinct real roots if $T^2 - 4D > 0$
 - ◊ 2 complex conjugate roots if $T^2 - 4D < 0$
 - ◊ Double real root if $T^2 - 4D = 0$

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Example

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$$

- $p(\lambda) = \lambda^2 - 2\lambda + 5$.
- Eigenvalues: $\lambda = 1 + 2i$ and $\bar{\lambda} = 1 - 2i$

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$$\lambda = 1 + 2i$$

- $A - \lambda I = \begin{pmatrix} -6 - 2i & 20 \\ -2 & 6 - 2i \end{pmatrix}$.
- Eigenvector: $\mathbf{w} = \begin{pmatrix} 3 - i \\ 1 \end{pmatrix}$

Return

Example

- Complex Solutions

$$\mathbf{z}(t) = e^{\lambda t} \mathbf{w} = e^{(1+2i)t} \begin{pmatrix} 3-i \\ 1 \end{pmatrix}$$

$$\bar{\mathbf{z}}(t) = e^{\bar{\lambda}t} \bar{\mathbf{w}} = e^{(1-2i)t} \begin{pmatrix} 3+i \\ 1 \end{pmatrix}$$

- Real Solutions

$$\mathbf{x}(t) = \operatorname{Re}(\mathbf{z}(t)) = e^t \begin{pmatrix} 3 \cos 2t + \sin 2t \\ \cos 2t \end{pmatrix}$$

$$\mathbf{y}(t) = \operatorname{Im}(\mathbf{z}(t)) = e^t \begin{pmatrix} 3 \sin 2t - \cos 2t \\ \sin 2t \end{pmatrix}$$

Return

Example

 λ

Initial Value Problem

Solve

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} -5 & 20 \\ -2 & 7 \end{pmatrix}$$

with the initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 5 \\ 3 \end{pmatrix}.$$

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Fundamental set of solutions

Initial Value Problem

Solution is

$$\begin{aligned} \mathbf{u}(t) &= 3e^t \begin{pmatrix} 3 \cos 2t + \sin 2t \\ \cos 2t \end{pmatrix} \\ &\quad + 4e^t \begin{pmatrix} 3 \sin 2t - \cos 2t \\ \sin 2t \end{pmatrix} \\ &= e^t \begin{pmatrix} 5 \cos 2t + 15 \sin 2t \\ 3 \cos 2t + 4 \sin 2t \end{pmatrix} \end{aligned}$$

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Real solutions

Summary — Complex Eigenvalues

Suppose A is a real 2×2 matrix with

- complex conjugate eigenvalues λ and $\bar{\lambda}$, and
- associated nonzero eigenvectors \mathbf{w} and $\bar{\mathbf{w}}$.

Then

- $\mathbf{z}(t) = e^{\lambda t} \mathbf{w}$ and $\bar{\mathbf{z}}(t) = e^{\bar{\lambda} t} \bar{\mathbf{w}}$ form a complex valued fundamental set of solutions, and
- $\mathbf{x}(t) = \operatorname{Re}(\mathbf{z}(t))$ and $\mathbf{y}(t) = \operatorname{Im}(\mathbf{z}(t))$ form a real valued fundamental set of solutions.

[Return](#)

