Math 211

Lecture #27

Phase Plane Portraits

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Double Real Root

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} = \frac{T}{2}.$$

- $T^2 - 4D = 0$
- First possibility:
  - ◇ Eigenspace has dimension 2: $\Leftrightarrow A = \lambda I$.
  - ◇ Every vector is an eigenvector. Every solution has the form

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}.$$

Example

$$\mathbf{x}' = A\mathbf{x} \quad \text{where} \quad A = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix}$$

- $p(\lambda) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2; \quad \lambda = -2$
- $A - \lambda I = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}; \quad \mathbf{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$
- Eigenspace has dimension 1, with basis $\mathbf{v}_1$.
- One solution:

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1 = e^{-2t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Return

Double Real Root

- Suppose the eigenspace has dimension 1.
  - Spanned by the eigenvector $\mathbf{v}_1 \neq 0$
  - Standard procedure gives only one solution,

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1.$$

- Look for a second solution of the form

$$\mathbf{x}_2(t) = e^{\lambda t}[\mathbf{v}_2 + t\mathbf{v}_1]$$

- Second solution of the form

$$\begin{aligned}\mathbf{x}_2(t) &= e^{\lambda t}[\mathbf{v}_2 + t\mathbf{v}_1] \\ \mathbf{x}_2' &= e^{\lambda t}[(\mathbf{v}_1 + \lambda\mathbf{v}_2) + \lambda t\mathbf{v}_1] \\ A\mathbf{x}_2 &= e^{\lambda t}[A\mathbf{v}_2 + tA\mathbf{v}_1]\end{aligned}$$

- $\mathbf{x}_2' = A\mathbf{x}_2 \Leftrightarrow$

$$A\mathbf{v}_1 = \lambda\mathbf{v}_1 \quad \text{and} \quad A\mathbf{v}_2 = \mathbf{v}_1 + \lambda\mathbf{v}_2.$$

- Need:
  - ◇ $\mathbf{v}_1$ to be an eigenvector.
  - ◇ $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$.

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Procedure in Degenerate Planar Case

- Find the (only) eigenvalue λ_1.
- Find an eigenvector $\mathbf{v}_1 \neq \mathbf{0}$.
- Find $\mathbf{v}_2$ with $(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1$.
  - Start with any vector $\mathbf{w}$ not a multiple of $\mathbf{v}_1$
  - $(A - \lambda I)\mathbf{w} = a\mathbf{v}_1$ with $a \neq 0$.
  - Set $\mathbf{v}_2 = \frac{1}{a}\mathbf{w}$.
- $\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1$ and $\mathbf{x}_2(t) = e^{\lambda t}[\mathbf{v}_2 + t\mathbf{v}_1]$.

Example (cont.)

- Start with

$$\mathbf{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$(A - \lambda I)\mathbf{w} = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} = -\mathbf{v}_1$$

- $\mathbf{v}_2 = -\mathbf{w} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

- Fundamental set of solutions:

$$\mathbf{x}_1(t) = e^{\lambda t}\mathbf{v}_1 = e^{-2t}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\begin{aligned} \mathbf{x}_2(t) &= e^{\lambda t}[\mathbf{v}_2 + t\mathbf{v}_1] \\ &= e^{-2t}\left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -3 \\ 1 \end{pmatrix}\right] \\ &= e^{-2t}\begin{pmatrix} -1-3t \\ t \end{pmatrix}. \end{aligned}$$

Planar System $\mathbf{x}' = A\mathbf{x}$

- Equilibrium points for the system
  - Set of equilibrium points equals $\mathrm{null}(A)$.
  - A nonsingular $\Rightarrow$ only equilibrium point is $\mathbf{0}$.
- Can we list the types of all possible equilibrium points for planar linear systems?
  - Six most important cases.
  - Look at solution curves in the phase plane.

Distinct Real Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D > 0$.

$$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} < \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$

- Eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. General solution

$$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$

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Exponential Solutions

$$\mathbf{x}(t) = Ce^{\lambda t}\mathbf{v}$$

- The solution curve is a straight half-line through $C\mathbf{v}$. Sometimes called *half-line* solutions.
- If $\lambda > 0$ the solution starts at $\mathbf{0}$ for $t = -\infty$, and tends to ∞ as $t \to \infty$. *Unstable solution*
- If $\lambda < 0$ the solution starts at ∞ for $t = -\infty$, and tends to $\mathbf{0}$ as $t \to \infty$. *Stable solution*

Saddle Point

- $\lambda_1 < 0 < \lambda_2$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$
- Two stable exponential solutions $(C_2 = 0)$
- Two unstable exponential solutions $(C_1 = 0)$.
- $C_1 \neq 0$ and $C_2 \neq 0$.
  - As $t \to \infty$, $\mathbf{x}(t) \to \infty$, approaching the half-line through $C_2\mathbf{v}_2$.
  - As $t \to -\infty$, $\mathbf{x}(t) \to \infty$, approaching the half-line through $C_2\mathbf{v}_1$.

Nodal Sink

- $\lambda_1 < \lambda_2 < 0$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$
- Four stable exponential solutions.
- All solutions $\to \mathbf{0}$ as $t \to \infty$. (Stable)
  - ◇ Tangent to $C_2\mathbf{v}_2$ if $C_2 \neq 0$.
- All solutions $\to \infty$ as $t \to -\infty$.
  - ◇ $\|$ to the half line through $C_1\mathbf{v}_1$ if $C_1 \neq 0$.

Nodal Source

- $0 < \lambda_1 < \lambda_2$
- General solution $\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$
- Four unstable exponential solutions.
- All solutions $\to \mathbf{0}$ as $t \to -\infty$.
  - ◇ Tangent to $C_1\mathbf{v}_1$ if $C_1 \neq 0$.
- All solutions $\to \infty$ as $t \to \infty$. (Unstable)
  - ◇ $\|$ to the half line through $C_2\mathbf{v}_2$ if $C_2 \neq 0$.

Complex Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D < 0$

$$\lambda = \alpha + i\beta \quad \text{and} \quad \overline{\lambda} = \alpha - i\beta.$$

- Eigenvector $\mathbf{w} = \mathbf{v}_1 + i\mathbf{v}_2$ associated to λ.
- Complex solutions

$$\mathbf{z}(t) = e^{\lambda t}\mathbf{w} = e^{t(\alpha + i\beta)}[\mathbf{v}_1 + i\mathbf{v}_2]$$

$$\overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}} = e^{t(\alpha - i\beta)}[\mathbf{v}_1 - i\mathbf{v}_2]$$

- Real solutions

$$\mathbf{x}_1(t) = \mathrm{Re}(\mathbf{z}(t)) = e^{\alpha t}[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2]$$

$$\mathbf{x}_2(t) = \mathrm{Im}(\mathbf{z}(t)) = e^{\alpha t}[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]$$

- General solution

$$\mathbf{x}(t) = C_1 e^{\alpha t}[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2] + C_2 e^{\alpha t}[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]$$

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Center

- $\alpha = \text{Re}(\lambda) = 0$
- General real solution

$$\mathbf{x}(t) = C_1[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2] + C_2[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]$$

- Every solution is periodic with period $T = 2\pi/\beta$.
- All solution curves are ellipses.

Return

Spiral Sink

- $\alpha = \mathrm{Re}(\lambda) < 0$
- General real solution

$$\mathbf{x}(t) = C_1 e^{\alpha t}[\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2]$$
$$+ C_2 e^{\alpha t}[\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2]$$

- All solutions spiral into $\mathbf{0}$ as $t \to \infty$.

Return

Spiral Source

- $\alpha = \text{Re}(\lambda) > 0$
- General real solution

$$\mathbf{x}(t) = C_1 e^{\alpha t}[\cos\beta t \cdot \mathbf{v}_1 - \sin\beta t \cdot \mathbf{v}_2]$$
$$+ C_2 e^{\alpha t}[\sin\beta t \cdot \mathbf{v}_1 + \cos\beta t \cdot \mathbf{v}_2]$$

- All solutions spiral into $\mathbf{0}$ as $t \to -\infty$.

Planar Systems

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- Char. polynomial $p(\lambda) = \lambda^2 - T\lambda + D$.
- Eigenvalues

$$\lambda_1,\ \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- $\lambda_1 \,\&\, \lambda_2$ are the roots of $p(\lambda)$, so

$$\begin{aligned} p(\lambda) &= (\lambda - \lambda_1)(\lambda - \lambda_2) \\ &= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \end{aligned}$$

- $T = \lambda_1 + \lambda_2$ and $D = \lambda_1\lambda_2$.
- Duality between $(\lambda_1,\ \lambda_2)$ and (T, D).
- Represent systems by location of (T, D) in the TD-plane.

Trace-Determinant Plane

- $T^2 - 4D > 0$
  - ◇ $\Rightarrow$ distinct real eigenvalues $\lambda_1 \,\&\, \lambda_2$
  - ◇ $D = \lambda_1\lambda_2 < 0 \Rightarrow$ Saddle point.
  - ◇ $D = \lambda_1\lambda_2 > 0 \Rightarrow$ Eigenvalues have the same sign.
    - ⋆ $T = \lambda_1 + \lambda_2 > 0 \Rightarrow$ Nodal source.
    - ⋆ $T = \lambda_1 + \lambda_2 < 0 \Rightarrow$ Nodal sink.

- $T^2 - 4D < 0 \Rightarrow$ complex eigenvalues

$$\lambda = \alpha + i\beta \quad \text{and} \quad \overline{\lambda} = \alpha - i\beta.$$

◇ $T = \lambda + \overline{\lambda} = 2\alpha > 0 \Rightarrow$ Spiral source.

◇ $T = \lambda + \overline{\lambda} = 2\alpha < 0 \Rightarrow$ Spiral sink.

◇ $T = \lambda + \overline{\lambda} = 2\alpha = 0 \Rightarrow$ Center.

Types of Equilibrium Points

- *Generic* types
  - Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - All occupy large open subsets of the trace-determinant plane.
- *Nongeneric* types
  - Center and eight others. Occupy pieces of the boundaries.

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