Math 211

Lecture #28

Higher Dimensional Systems

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Planar Systems

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

- Char. polynomial $p(\lambda) = \lambda^2 - T\lambda + D$.
- Eigenvalues

$$\lambda_1,\ \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$

- $\lambda_1 \,\&\, \lambda_2$ are the roots of $p(\lambda)$, so

$$\begin{aligned} p(\lambda) &= \lambda^2 - T\lambda + D \\ &= (\lambda - \lambda_1)(\lambda - \lambda_2) \\ &= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \end{aligned}$$

- $T = \lambda_1 + \lambda_2$ and $D = \lambda_1\lambda_2$.
- Duality between $(\lambda_1,\ \lambda_2)$ and (T, D).
- Represent systems by location of (T, D) in the TD-plane.

Trace-Determinant Plane

- $T^2 - 4D > 0$
  - $\Rightarrow$ distinct real eigenvalues $\lambda_1 \,\&\, \lambda_2$
  - $D = \lambda_1 \lambda_2 < 0 \Rightarrow$ Saddle point.
  - $D = \lambda_1 \lambda_2 > 0 \Rightarrow$ Eigenvalues have the same sign.
    - $T = \lambda_1 + \lambda_2 > 0 \Rightarrow$ Nodal source.
    - $T = \lambda_1 + \lambda_2 < 0 \Rightarrow$ Nodal sink.

- $T^2 - 4D < 0 \Rightarrow$ complex eigenvalues

$$\lambda = \alpha + i\beta \quad \text{and} \quad \overline{\lambda} = \alpha - i\beta.$$

- ◇ $T = \lambda + \overline{\lambda} = 2\alpha > 0 \Rightarrow$ Spiral source.
- ◇ $T = \lambda + \overline{\lambda} = 2\alpha < 0 \Rightarrow$ Spiral sink.
- ◇ $T = \lambda + \overline{\lambda} = 2\alpha = 0 \Rightarrow$ Center.

Types of Equilibrium Points

- *Generic* types
  - ◇ Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - ◇ All occupy large open subsets of the trace-determinant plane.
- *Nongeneric* types
  - ◇ Center and eight others. Occupy pieces of the boundaries.

Higher Dimensional Systems

$$\mathbf{x}' = A\mathbf{x}$$

- A is a real $n \times n$ matrix.
- If λ is an eigenvalue and $\mathbf{v} \neq 0$ is an associated eigenvector, then $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ is a solution.
- Much like the planar case, but now we need n linearly independent solutions.

Return

Proposition: Suppose that $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of A, and that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are associated nonzero eigenvectors. Then $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent.

Theorem: Suppose the $n \times n$ real matrix A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, and that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are associated nonzero eigenvectors. Then the exponential solutions $\mathbf{x}_i(t) = e^{\lambda_i t}\mathbf{v}_i$, $1 \le i \le n$ form a fundamental set of solutions for the system
$\mathbf{x}' = A\mathbf{x}$.

Return

Example

$$A = \begin{pmatrix} 17 & -30 & -8 \\ 16 & -29 & -8 \\ -12 & 24 & 7 \end{pmatrix}$$

- Use MATLAB.

Return

Complex Eigenvalues

A a real $n \times n$ matrix with a complex eigenvalue λ and associate eigenvector $\mathbf{w}$.

- $\Rightarrow \overline{\lambda}$ is an eigenvalue and $\overline{\mathbf{w}}$ is an associated nonzero eigenvector.
- Complex valued solutions: $$\mathbf{z}(t) = e^{\lambda t}\mathbf{w}$$ $$\overline{\mathbf{z}}(t) = e^{\overline{\lambda} t}\overline{\mathbf{w}}.$$
- Real solutions: $$\mathbf{x}(t) = \mathrm{Re}(\mathbf{z}(t))$$ $$\mathbf{y}(t) = \mathrm{Im}(\mathbf{z}(t)).$$

Example

$$A = \begin{pmatrix} 21 & 10 & 4 \\ -70 & -31 & -10 \\ 30 & 10 & -1 \end{pmatrix}$$

- The theorem applies if some of the eigenvalues are complex and we replace complex conjugate pairs of solutions by their real and imaginary parts.

Repeated Eigenvalues – Example 1

$$A = \begin{pmatrix} -5 & -10 & 6 \\ 8 & 19 & -12 \\ 12 & 30 & -19 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- $\lambda_1 = -3$
  - ◇ Eigenspace has dimension 1 $\Rightarrow$ one exponential solution

$$\mathbf{x}_1(t) = e^{-3t}(-1/3, 2/3, 1)^T$$

- $\lambda_2 = -1$
  - Eigenspace has dimension 2 $\Rightarrow$ two linearly independent exponential solutions

$$\mathbf{x}_2(t) = e^{-t}\begin{pmatrix} -5/2 \\ 1 \\ 0 \end{pmatrix} \quad \& \quad \mathbf{x}_3(t) = e^{-t}\begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix}$$

- $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are a fundamental set of solutions.

Repeated Eigenvalues – Example 2

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- $\lambda_1 = -3$
  - ◇ Eigenspace has dimension 1 $\Rightarrow$ one exponential solution

$$\mathbf{x}_1(t) = e^{-3t}(-1/2, 3/2, 1)^T$$

- $\lambda_2 = -1$
  - ◇ Eigenspace has dimension 1 $\Rightarrow$ only one exponential solution

$$\mathbf{x}_2(t) = e^{-t} \begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix}$$

- Need a third solution.
- Need a new idea.

Multiplicities

A an $n \times n$ matrix

- Distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.
- The characteristic polynomial is

$$p(\lambda) = (\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2} \cdot \ldots \cdot (\lambda - \lambda_k)^{q_k}.$$

- The *algebraic multiplicity* of λ_j is q_j.
- The *geometric multiplicity* of λ_j is d_j, the dimension of the eigenspace of λ_j.

- We always have:
  - $q_1 + q_2 + \cdots + q_k = n$.
  - $1 \leq d_j \leq q_j$.
  - There are d_j linearly independent exponential solutions corresponding to λ_j.
  - If $d_j = q_j$ for all j we have n linearly independent solutions.

Examples

- In both $p(\lambda) = (\lambda + 1)^2(\lambda + 3)$.
- In both $\lambda_2 = -1$ has algebraic multiplicity 2.
- In Ex. 1 $\lambda_2 = -1$ has geom. multiplicity 2.
- In Ex. 2 $\lambda_2 = -1$ has geom. multiplicity 1.
- Problems arise when $d_j < q_j$.

New Approach

- $D = 1 : x' = ax$
  - Solution $x(t) = Ce^{at}$.
- $D > 1 : \mathbf{x}' = A\mathbf{x}$
  - Tried $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$.
  - Why not $\mathbf{x}(t) = e^{tA}\mathbf{v}$?
- But what is e^{tA}?

Exponential of a Matrix

Definition: The *exponential* of the $n \times n$ matrix A is the $n \times n$ matrix

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$
$$= \sum_0^\infty \frac{1}{n!}A^n.$$

Examples

- $A = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$

$$e^A = \begin{pmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{pmatrix}.$$

- $e^{\lambda I} = e^{\lambda} I.$
- $e^{0I} = I.$

Return

Properties

- A commutes with e^A,

$$Ae^A = e^A A.$$

- If A and B commute (i.e., $AB = BA$), then

$$e^{A+B} = e^A \cdot e^B.$$

- The inverse of e^A is e^{-A}.
- $\frac{d}{dt} e^{tA} = Ae^{tA}$.

Important Fact

The solution to the intial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}$$

is

$$\mathbf{x}(t) = e^{tA}\mathbf{v}.$$

- However computing e^{tA} is not easy.