

# Math 211

Lecture #28

The Exponential of a Matrix

March 30, 2001

# Multiplicities

$A$  an  $n \times n$  matrix

- Distinct eigenvalues  $\lambda_1, \dots, \lambda_k$ .
- The characteristic polynomial is

$$p(\lambda) = (\lambda - \lambda_1)^{q_1} (\lambda - \lambda_2)^{q_2} \cdot \dots \cdot (\lambda - \lambda_k)^{q_k}.$$

- The *algebraic multiplicity* of  $\lambda_j$  is  $q_j$ .
- The *geometric multiplicity* of  $\lambda_j$  is  $d_j$ , the dimension of the eigenspace of  $\lambda_j$ .

- We always have:
  - ◇  $q_1 + q_2 + \cdots + q_k = n$ .
  - ◇  $1 \leq d_j \leq q_j$ .
  - ◇ There are  $d_j$  linearly independent exponential solutions corresponding to  $\lambda_j$ .
  - ◇ If  $d_j = q_j$  for all  $j$  we have  $n$  linearly independent solutions.
- If  $d_j < q_j$  we have trouble.

## New Approach

- $D = 1 : x' = ax$ 
  - ◇ Solution  $x(t) = Ce^{at}$ .
- $D > 1 : \mathbf{x}' = A\mathbf{x}$ 
  - ◇ Tried  $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$ .
  - ◇ Why not  $\mathbf{x}(t) = e^{tA}\mathbf{v}$ ?
- But what is  $e^{tA}$ ?

## Exponential of a Matrix

**Definition:** The *exponential* of the  $n \times n$  matrix  $A$  is the  $n \times n$  matrix

$$\begin{aligned} e^A &= I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \dots \\ &= \sum_0^{\infty} \frac{1}{n!}A^n. \end{aligned}$$

## Examples

- $A = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}$

$$e^A = \begin{pmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{pmatrix}.$$

- $e^{\lambda I} = e^\lambda I.$

- $e^{0I} = I.$

# Properties

- $A$  commutes with  $e^A$ ,

$$Ae^A = e^A A.$$

- If  $A$  and  $B$  commute (i.e.,  $AB = BA$ ), then

$$e^{A+B} = e^A \cdot e^B.$$

- The inverse of  $e^A$  is  $e^{-A}$ .

- $\frac{d}{dt} e^{tA} = Ae^{tA}$ .

## Important Fact

The solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}$$

is

$$\mathbf{x}(t) = e^{tA}\mathbf{v}.$$

- However computing  $e^{tA}$  is not easy.

## Key to Computing $e^{tA}$ or $e^{tA}\mathbf{v}$

$A$  an  $n \times n$  matrix, and  $\lambda$  a number (an eigenvalue).

- $A = \lambda I + (A - \lambda I)$  ;  $\lambda I$  &  $A - \lambda I$  commute.

$$\begin{aligned}
 e^{tA} &= e^{t[\lambda I + (A - \lambda I)]} \\
 &= e^{t\lambda I} \cdot e^{t(A - \lambda I)} \\
 &= e^{\lambda t} \cdot e^{t(A - \lambda I)} \\
 &= e^{\lambda t} \cdot [I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \dots]
 \end{aligned}$$

Return

$e^{t\lambda I}$

$e^{t(A - \lambda I)}$

Solution

## $e^{tA}\mathbf{v}$ , $\mathbf{v}$ an Eigenvector

Let  $\lambda$  be an eigenvalue and  $\mathbf{v}$  an associated eigenvector.  $\Leftrightarrow (A - \lambda I)\mathbf{v} = \mathbf{0}$ .

$$\begin{aligned}
 e^{tA}\mathbf{v} &= e^{\lambda t} \cdot e^{t(A-\lambda I)}\mathbf{v} \\
 &= e^{\lambda t} \left[ I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \dots \right] \mathbf{v} \\
 &= e^{\lambda t} \left[ \mathbf{v} + t(A - \lambda I)\mathbf{v} + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{v} + \dots \right] \\
 &= e^{\lambda t}\mathbf{v}
 \end{aligned}$$

- The infinite series truncates, so we can compute  $e^{tA}\mathbf{v}$ .

# Matrices with One Eigenvalue

$A$  an  $n \times n$  matrix with characteristic polynomial

$$p(\lambda) = (\lambda - \lambda_1)^n.$$

- *Cayley-Hamilton Theorem:* If  $p(\lambda)$  is the characteristic polynomial of the matrix  $A$  then  $p(A) = 0I$ .
- In our case

$$(A - \lambda_1 I)^n = 0I.$$

- $(A - \lambda_1 I)^n = 0I \Rightarrow$

$$\begin{aligned} e^{tA} &= e^{\lambda_1 t} \cdot [I + t(A - \lambda_1 I) \\ &\quad + \frac{t^2}{2!} (A - \lambda_1 I)^2 + \dots \\ &\quad + \frac{t^{n-1}}{(n-1)!} (A - \lambda_1 I)^{n-1}] \end{aligned}$$

- The infinite series truncates, so we can compute  $e^{tA}$ .

## Example 1

$$A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 2)^2$ .

$$A + 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \quad (A + 2I)^2 = 0I$$

$$\begin{aligned} e^{tA} &= e^{-2t} [I + t(A + 2I)] \\ &= e^{-2t} \begin{pmatrix} 1 - t & t \\ -t & 1 + t \end{pmatrix}. \end{aligned}$$

## Example 2

$$A = \begin{pmatrix} 0 & -9 & 27 \\ -2 & 3 & -18 \\ -1 & 3 & -12 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)^3$ .  $(A + 3I)^2 = 0I$ .

$$e^{tA} = e^{-3t} [I + t(A + 3I)]$$

$$= e^{-3t} \begin{pmatrix} 1 + 3t & -9t & 27t \\ -2t & 1 + 6t & -18t \\ -t & 3t & 1 - 9t \end{pmatrix}.$$

## Example 3

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix}$$

- Earlier example.
- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- Distinct eigenvalues  $\lambda_1 = -3$  &  $\lambda_2 = -1$
- Different from previous two examples.

- $\lambda_1 = -3$ 
  - ◇ Eigenspace has dimension 1  $\Rightarrow$  one exponential solution

$$\begin{aligned}\mathbf{x}_1(t) &= e^{\lambda_1 t} \mathbf{v}_1 \\ &= e^{-3t} \begin{pmatrix} -1/2 \\ 3/2 \\ 1 \end{pmatrix}\end{aligned}$$

- $\lambda_2 = -1$ 
  - ◇ Eigenspace has dimension 1  $\Rightarrow$  only one exponential solution

$$\begin{aligned}\mathbf{x}_2(t) &= e^{\lambda_2 t} \mathbf{v}_2 \\ &= e^{-t} \begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix}\end{aligned}$$

- ◇ However,  $\text{null}((A - \lambda_2 I)^2)$  has dimension 2.

- If  $\mathbf{v} \in \text{null}((A - \lambda_2 I)^2)$  then

$$\begin{aligned} e^{tA}\mathbf{v} &= e^{\lambda_2 t} [I + t(A - \lambda_2 I) \\ &\quad + \frac{t^2}{2!} (A - \lambda_2 I)^2 + \cdots] \mathbf{v} \\ &= e^{\lambda_2 t} [\mathbf{v} + t(A - \lambda_2 I)\mathbf{v} \\ &\quad + \frac{t^2}{2!} (A - \lambda_2 I)^2 \mathbf{v} + \cdots] \\ &= e^{\lambda_2 t} [\mathbf{v} + t(A - \lambda_2 I)\mathbf{v}]. \end{aligned}$$

- $\text{null}(A + I)^2$  has basis

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Third solution:

$$\mathbf{x}_3(t) = e^{tA}\mathbf{v}_3 = e^{-t}[\mathbf{v}_3 + t(A + I)\mathbf{v}_3]$$

$$\begin{aligned}\mathbf{x}_3(t) &= e^{-t}[\mathbf{v}_3 + t(A + I)\mathbf{v}_3] \\ &= e^{-t}[\mathbf{v}_3 - 4t\mathbf{v}_2] \\ &= e^{-t} \begin{pmatrix} 1 + 2t \\ -4t \\ -4t \end{pmatrix}.\end{aligned}$$

## Summary

- In Examples 1 & 2 the matrix has one eigenvalue.
  - ◇ The series for  $e^{t(A-\lambda I)}$  truncates to a finite sum.
- In Example 3 the matrix had two eigenvalues.
  - ◇ The series for  $e^{t(A-\lambda_2 I)}$  does not truncate.
  - ◇ The series for  $e^{t(A-\lambda_2 I)}\mathbf{v}$  does **truncate** if  $(A - \lambda_2 I)^2\mathbf{v} = \mathbf{0}$ .

## Generalized Eigenvectors

**Definition:** If  $\lambda$  is an eigenvalue of  $A$  and  $(A - \lambda I)^p \mathbf{v} = \mathbf{0}$  for some integer  $p \geq 1$ , then  $\mathbf{v}$  is called a *generalized eigenvector* associated with  $\lambda$ .

- The **series** for  $e^{t(A-\lambda I)} \mathbf{v}$  truncates to a finite sum if  $\mathbf{v}$  is a generalized eigenvector associated with  $\lambda$ .
- We can compute  $e^{tA} \mathbf{v}$ .

**Theorem:** If  $\lambda$  is an eigenvalue of  $A$  with algebraic multiplicity  $q$ , then there is an integer  $p \leq q$  such that  $\text{null}((A - \lambda I)^p)$  has dimension  $q$ .

- For each generalized eigenvector  $\mathbf{v}$  we can compute  $e^{tA}\mathbf{v}$ .
- We can find  $q$  linearly independent solutions associated with the eigenvalue  $\lambda$ .

## Procedure for $\lambda$ of algebraic multiplicity $q$

To find  $q$  linearly independent solutions associated with an eigenvalue  $\lambda$  of algebraic multiplicity  $q$ .

- Find the smallest integer  $p$  such that  $\text{null}((A - \lambda I)^p)$  has dimension  $q$ .
- Find a basis  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_q$  of  $\text{null}((A - \lambda I)^p)$ .
- For  $j = 1, 2, \dots, q$

$$\mathbf{x}_j(t) = e^{tA} \mathbf{v}_j.$$

- For  $j = 1, 2, \dots, q$

$$\begin{aligned}\mathbf{x}_j(t) &= e^{tA} \mathbf{v}_j \\ &= e^{\lambda t} [\mathbf{v}_j + t(A - \lambda I) \mathbf{v}_j \\ &\quad + \frac{t^2}{2!} (A - \lambda I)^2 \mathbf{v}_j + \dots \\ &\quad + \frac{t^{p-1}}{(p-1)!} (A - \lambda I)^{p-1} \mathbf{v}_j]\end{aligned}$$

- ◇ The series may truncate before  $p - 1$ .

## Example

- Use MATLAB.

## Procedure for a Complex Eigenvalue

If  $\lambda$  is complex of algebraic multiplicity  $q$ . Then  $\bar{\lambda}$  also has multiplicity  $q$ .

- Find the smallest integer  $p$  such that  $\text{null}((A - \lambda I)^p)$  has dimension  $q$ .
- Find a basis  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_q$  of  $\text{null}((A - \lambda I)^p)$ .
- For  $j = 1, 2, \dots, q$

$$\mathbf{z}_j(t) = e^{tA} \mathbf{w}_j.$$

- For  $j = 1, 2, \dots, q$

$$\begin{aligned}\mathbf{z}_j(t) &= e^{\lambda t} [\mathbf{w}_j + t(A - \lambda I)\mathbf{w}_j \\ &\quad + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{w}_j + \dots \\ &\quad + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}\mathbf{w}_j]\end{aligned}$$

- For  $j = 1, 2, \dots, q$  set

$$\mathbf{x}_j(t) = \operatorname{Re}(\mathbf{z}_j(t)) \quad \text{and}$$

$$\mathbf{y}_j(t) = \operatorname{Im}(\mathbf{z}_j(t)).$$