Math 211

Lecture #30

The Exponential of a Matrix
Stability of Solutions

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Important Fact

The solution to the initial value problem

$$\mathbf{x}' = A\mathbf{x} \quad \text{with} \quad \mathbf{x}(0) = \mathbf{v}$$

is

$$\mathbf{x}(t) = e^{tA}\mathbf{v}.$$

- However computing e^{tA} is not easy.

Return

Key to Computing e^{tA} or $e^{tA}\mathbf{v}$

A an $n \times n$ matrix, and λ a number (an eigenvalue).

- $A = \lambda I + (A - \lambda I)$; λI & $A - \lambda I$ commute.

$$\begin{aligned} e^{tA} &= e^{t[\lambda I + (A - \lambda I)]} \\ &= e^{t\lambda I} \cdot e^{t(A-\lambda I)} \\ &= e^{\lambda t} \cdot e^{t(A-\lambda I)} \\ &= e^{\lambda t} \cdot \left[I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots \right] \end{aligned}$$

$e^{tA}\mathbf{v}$, $\mathbf{v}$ an Eigenvector

Let λ be an eigenvalue and $\mathbf{v}$ an associated eigenvector. $\Leftrightarrow (A - \lambda I)\mathbf{v} = \mathbf{0}$.

$$
\begin{aligned}
e^{tA}\mathbf{v} &= e^{\lambda t} \cdot e^{t(A-\lambda I)}\mathbf{v} \\
&= e^{\lambda t}[I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots]\mathbf{v} \\
&= e^{\lambda t}[\mathbf{v} + t(A - \lambda I)\mathbf{v} + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{v} + \cdots] \\
&= e^{\lambda t}\mathbf{v}
\end{aligned}
$$

- The infinite series truncates, so we can compute $e^{tA}\mathbf{v}$.

Matrices with One Eigenvalue

A an $n \times n$ matrix with characteristic polynomial

$$p(\lambda) = (\lambda - \lambda_1)^n.$$

- *Cayley-Hamilton Theorem:* If $p(\lambda)$ is the characteristic polynomial of the matrix A then $p(A) = 0I$.
- In our case

$$(A - \lambda_1 I)^n = 0I.$$

- $(A - \lambda_1 I)^n = 0I \quad \Rightarrow$

$$e^{tA} = e^{\lambda_1 t} \cdot [I + t(A - \lambda_1 I) + \frac{t^2}{2!}(A - \lambda_1 I)^2 + \cdots + \frac{t^{n-1}}{(n-1)!}(A - \lambda_1 I)^{n-1}]$$

- The infinite series truncates, so we can compute e^{tA}.

Example 1

$$A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 2)^2.$

$$A + 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \qquad (A + 2I)^2 = 0I$$

$$\begin{aligned} e^{tA} &= e^{-2t}[I + t(A + 2I)] \\ &= e^{-2t} \begin{pmatrix} 1 - t & t \\ -t & 1 + t \end{pmatrix}. \end{aligned}$$

Example 2

$$A = \begin{pmatrix} 0 & -9 & 27 \\ -2 & 3 & -18 \\ -1 & 3 & -12 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)^3$. $\quad (A + 3I)^2 = 0I$.

$$e^{tA} = e^{-3t}[I + t(A + 3I)]$$

$$= e^{-3t} \begin{pmatrix} 1 + 3t & -9t & 27t \\ -2t & 1 + 6t & -18t \\ -t & 3t & 1 - 9t \end{pmatrix}.$$

Example 3

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- Distinct eigenvalues $\lambda_1 = -3$ & $\lambda_2 = -1$
- Different from previous two examples.

Return

- $\lambda_1 = -3$
  - ◇ Algebraic multiplicity $q_1 = 1$.
  - ◇ Eigenspace has dimension 1
    $\Rightarrow$ geometric multiplicity $d_1 = 1$
    $\Rightarrow$ one exponential solution

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1$$
$$= e^{-3t}\begin{pmatrix} -1/2 \\ 3/2 \\ 1 \end{pmatrix}$$

- $\lambda_2 = -1$
  - ◇ Algebraic multiplicity $q_2 = 2$.
  - ◇ Eigenspace has dimension 1
    $\Rightarrow$ geometric multiplicity $d_2 = 1$
    $\Rightarrow$ only one exponential solution

$$\mathbf{x}_2(t) = e^{\lambda_2 t}\mathbf{v}_2$$
$$= e^{-t}\begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix}$$

  - ◇ However, $\mathrm{null}((A - \lambda_2 I)^2)$ has dimension 2.

Return

- If $\mathbf{v} \in \text{null}((A - \lambda_2 I)^2)$ then

$$\begin{aligned} e^{tA}\mathbf{v} &= e^{\lambda_2 t}[I + t(A - \lambda_2 I) \\ &\quad + \frac{t^2}{2!}(A - \lambda_2 I)^2 + \cdots]\mathbf{v} \\ &= e^{\lambda_2 t}[\mathbf{v} + t(A - \lambda_2 I)\mathbf{v} \\ &\quad + \frac{t^2}{2!}(A - \lambda_2 I)^2\mathbf{v} + \cdots] \\ &= e^{\lambda_2 t}[\mathbf{v} + t(A - \lambda_2 I)\mathbf{v}]. \end{aligned}$$

- $\text{null}(A+I)^2$ has basis

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

- Third solution:

$$\mathbf{x}_3(t) = e^{tA}\mathbf{v}_3 = e^{-t}[\mathbf{v}_3 + t(A+I)\mathbf{v}_3]$$

$$\begin{aligned}\mathbf{x}_3(t) &= e^{-t}[\mathbf{v}_3 + t(A+I)\mathbf{v}_3] \\ &= e^{-t}[\mathbf{v}_3 - 4t\mathbf{v}_2] \\ &= e^{-t}\begin{pmatrix} 1+2t \\ -4t \\ -4t \end{pmatrix}.\end{aligned}$$

- $\mathbf{x}_1$, $\mathbf{x}_2$, & $\mathbf{x}_3$ are a fundamental set of solutions.

Summary

- In Examples 1 & 2 the matrix has one eigenvalue.
  - ◇ The series for $e^{t(A-\lambda I)}$ truncates to a finite sum.
  - ◇ We can compute e^{tA}.
  - ◇ We can compute $e^{tA}\mathbf{v}$ for any vector $\mathbf{v}$.

- In Example 3 the matrix had two eigenvalues.
  - We cannot compute e^{tA} in this way.
  - The series for $e^{t(A-\lambda_2 I)}\mathbf{v}$ does truncate if $(A-\lambda_2 I)^2\mathbf{v} = \mathbf{0}$.
  - We can compute $e^{tA}\mathbf{v}$ for any vector $\mathbf{v}$ in $\text{null}((A-\lambda_2 I)^2)$.
  - We can compute $\mathbf{x}_j(t) = e^{tA}\mathbf{v}_j$ for $j = 1, 2, \& 3$.

Generalized Eigenvectors

Definition: If λ is an eigenvalue of A and $(A - \lambda I)^p \mathbf{v} = \mathbf{0}$ for some integer $p \geq 1$, then $\mathbf{v}$ is called a *generalized eigenvector* associated with λ.

- The series for $e^{t(A-\lambda I)}\mathbf{v}$ truncates to a finite sum if $\mathbf{v}$ is a generalized eigenvector associated with λ.
- We can compute $e^{tA}\mathbf{v}$ for any generalized eigenvector $\mathbf{v}$.

Theorem: If λ is an eigenvalue of A with algebraic multiplicity q, then there is an integer $p \leq q$ such that $\text{null}((A - \lambda I)^p)$ has dimension q.

- For each generalized eigenvector $\mathbf{v}$ we can compute $e^{tA}\mathbf{v}$.
- We can find q linearly independent solutions associated with the eigenvalue λ.
- $q_1 + q_2 + \cdots + q_k = n$ so we can find n linearly independent solutions.

Procedure for λ of algebraic multiplicity q

To find q linearly independent solutions associated with an eigenvalue λ of algebraic multiplicity q.

- Find the smallest integer p such that $\text{null}((A - \lambda I)^p)$ has dimension q.
- Find a basis $\mathbf{v}_1$, $\mathbf{v}_2$, . . . , $\mathbf{v}_q$ of $\text{null}((A - \lambda I)^p)$.
- For $j = 1, 2, \ldots, q$

$$\mathbf{x}_j(t) = e^{tA}\mathbf{v}_j.$$

Return

- For $j = 1, 2, \ldots, q$

$$\begin{aligned} \mathbf{x}_j(t) &= e^{tA}\mathbf{v}_j \\ &= e^{\lambda t}[\mathbf{v}_j + t(A - \lambda I)\mathbf{v}_j \\ &\quad + \frac{t^2}{2!}(A - \lambda I)^2\mathbf{v}_j + \cdots \\ &\quad + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}\mathbf{v}_j] \end{aligned}$$

  - ◇ The series may truncate before $p - 1$.

Example

- Use MATLAB.

Procedure for a Complex Eigenvalue

If λ is complex of algebraic multiplicity q. Then $\overline{\lambda}$ also has multiplicity q.

- Find the smallest integer p such that $\mathrm{null}((A - \lambda I)^p)$ has dimension q.
- Find a basis $\mathbf{w}_1$, $\mathbf{w}_2$, $\ldots$, $\mathbf{w}_q$ of $\mathrm{null}((A - \lambda I)^p)$.
- For $j = 1, 2, \ldots, q$

$$\mathbf{z}_j(t) = e^{tA}\mathbf{w}.$$

Return

- For $j = 1, 2, \ldots, q$

$$\begin{aligned}\mathbf{z}_j(t) = e^{\lambda t}[\mathbf{w}_j &+ t(A - \lambda I)\mathbf{w}_j \\ &+ \frac{t^2}{2!}(A - \lambda I)^2\mathbf{w}_j + \cdots \\ &+ \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}\mathbf{w}_j]\end{aligned}$$

- For $j = 1, 2, \ldots, q$ set

$$\mathbf{x}_j(t) = \mathrm{Re}(\mathbf{z}_j(t)) \quad \text{and}$$

$$\mathbf{y}_j(t) = \mathrm{Im}(\mathbf{z}_j(t)).$$