Math 211

Lecture #31

Stability of Solutions
Higher Order Equations

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Stability

Autonomous system $\mathbf{x}' = \mathbf{f}(\mathbf{x})$ with an equilibrium point at $\mathbf{x}_0$.

- Basic question: What happens to all solutions as $t \to \infty$?
- $\mathbf{x}_0$ is *stable* if for every $\epsilon > 0$ there is a $\delta > 0$ such that a solution $\mathbf{x}(t)$ with $|\mathbf{x}(0) - \mathbf{x}_0| < \delta$ $\Rightarrow$ $|\mathbf{x}(t) - \mathbf{x}_0| < \epsilon$ for all $t \geq 0$.

- $\mathbf{x}_0$ is *asymptotically stable* if it is stable and there is an $\eta > 0$ such that if $\mathbf{x}(t)$ is a solution with $|\mathbf{x}(0) - \mathbf{x}_0| < \eta$, then $\mathbf{x}(t) \to \mathbf{x}_0$ as $t \to \infty$.
  - $\mathbf{x}_0$ is called a *sink.*
- $\mathbf{x}_0$ is *unstable* if there is an $\epsilon > 0$ such that for any $\delta > 0$ there is a solution $\mathbf{x}(t)$ with $|\mathbf{x}(0) - \mathbf{x}_0| < \delta$ with the property that there are values of $t > 0$ such that $|\mathbf{x}(t) - \mathbf{x}_0| > \epsilon$.

Examples $D = 2$

- Sinks are asymptotically stable.
- Sources are unstable.
- Saddles are unstable.
- Centers are stable but not asymptotically stable.

Return

Theorem: Let A be an $n \times n$ real matrix.

- Suppose the real part of every eigenvalue of A is negative. Then $\mathbf{0}$ is an asymptotically stable equilibrium point for the system $\mathbf{x}' = A\mathbf{x}$.
- Suppose A has at least one eigenvalue with positive real part. Then $\mathbf{0}$ is an unstable equilibrium point for the system $\mathbf{x}' = A\mathbf{x}$.

Examples

- $D = 2$
- $T^2 - 4D = 0.$
  - $T < 0 \Rightarrow$ sink. $T > 0 \Rightarrow$ source.

- $\mathbf{y}' = A\mathbf{y}$,

$$A = \begin{pmatrix} -2 & -18 & -7 & -14 \\ 1 & 6 & 2 & 5 \\ 2 & 2 & -3 & 0 \\ -2 & -8 & -1 & -6 \end{pmatrix}.$$

  - ◇ A has eigenvalues -1, -2, & $-1 \pm i$.

Higher Order Equations

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

- Second order: $y'' + py' + qy = 0$.
- Equivalent system: $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

- A fundamental set of solutions for the system consists of two linearly independent solutions.

Linear Independence

Definition: Two functions $u(t)$ and $v(t)$ are *linearly independent* if neither is a constant multiple of the other.

- $\Leftrightarrow \begin{pmatrix} u \\ u' \end{pmatrix}$ & $\begin{pmatrix} v \\ v' \end{pmatrix}$ are linearly independent.

General Solution

Theorem: Suppose that $y_1(t)$ & $y_2(t)$ are linearly independent solutions to the equation

$$y'' + py' + qy = 0.$$

Then the general solution is

$$y(t) = C_1 y_1(t) + C_2 y_2(t).$$

Definition: A set of two linearly independent solutions is called a *fundamental set of solutions.*

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Solutions to $y'' + py' + qy = 0$.

- Equivalent system: $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

- Look for exponential solutions $y(t) = e^{\lambda t}$.
- *Characteristic equation:* $\lambda^2 + p\lambda + q = 0$.
- *Characteristic polynomial:* $\lambda^2 + p\lambda + q$.
- Same for the 2^{nd} order equation and the system.

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Real Roots

- If λ is a root to the characteristic polynomial then $y(t) = e^{\lambda t}$ is a solution.
- If λ is a root to the characteristic polynomial of multiplicity 2, then $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ are linearly independent solutions.

Complex Roots

- If $\lambda = \alpha + i\beta$ is a complex root of the characteristic equation , then so is $\overline{\lambda} = \alpha - i\beta$.
- A complex valued fundamental set of solutions is

$$z(t) = e^{\lambda t} \quad \text{and} \quad \overline{z}(t) = e^{\overline{\lambda} t}.$$

- A real valued fundamental set of solutions is

$$x(t) = e^{\alpha t} \cos \beta t \quad \text{and} \quad y(t) = e^{\alpha t} \sin \beta t.$$

Examples

- $y'' - 5y' + 6y = 0.$
- $y'' + 25y = 0.$
- $y'' + 4y' + 13y = 0.$

The Vibrating Spring

Newton's second law: $ma =$ total force.

- Forces acting:
  - Gravity mg.
  - Restoring force $R(x)$.
  - Damping force $D(v)$.
  - External force $F(t)$.

- Newton's law becomes

$$ma = mg + R(x) + D(v) + F(t)$$

- Hooke's law: $R(x) = -kx$. $k > 0$ is the *spring constant.*
- Spring-mass equilibrium $x_0 = mg/k$.
- Set $y = x - x_0$. Newton's law becomes

$$my'' = -ky + D(y') + F(t).$$

- Damping force $D(y') = -\mu y'$.
- Newton's law becomes

$$my'' = -ky - \mu y' + F(t), \quad \text{or}$$

$$my'' + \mu y' + ky = F(t), \quad \text{or}$$

$$y'' + \frac{\mu}{m} y' + \frac{k}{m} y = \frac{1}{m} F(t).$$