

# Math 211

Lecture #32

Higher Order Equations  
Harmonic Motion

April 6, 2001

## Solutions to $y'' + py' + qy = 0$ .

- Equivalent system:  $\mathbf{x}' = A\mathbf{x}$ , where

$$\mathbf{x} = \begin{pmatrix} y \\ y' \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

- Look for exponential solutions  $y(t) = e^{\lambda t}$ .
- *Characteristic equation:*  $\lambda^2 + p\lambda + q = 0$ .
- *Characteristic polynomial:*  $\lambda^2 + p\lambda + q$ .
- Same for the 2<sup>nd</sup> order equation and the system.
- Need two linearly independent solutions.

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## Real Roots

- If  $\lambda$  is a root to the characteristic polynomial then  $y(t) = e^{\lambda t}$  is a solution.
  - ◊  $y'' + 4y' + 3y = 0$
- If  $\lambda$  is a root to the characteristic polynomial of multiplicity 2, then  $y_1(t) = e^{\lambda t}$  and  $y_2(t) = te^{\lambda t}$  are linearly independent solutions.
  - ◊  $y'' + 4y' + 4y = 0$

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## Complex Roots

- If  $\lambda = \alpha + i\beta$  is a complex root of the characteristic equation, then so is  $\bar{\lambda} = \alpha - i\beta$ .
- A complex valued fundamental set of solutions is

$$z(t) = e^{\lambda t} \quad \text{and} \quad \bar{z}(t) = e^{\bar{\lambda}t}.$$

- A real valued fundamental set of solutions is

$$x(t) = e^{\alpha t} \cos \beta t \quad \text{and} \quad y(t) = e^{\alpha t} \sin \beta t.$$

$$\diamond y'' + 4y' + 8y = 0$$

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## Examples

- $y'' - 5y' + 6y = 0$ .
- $y'' + 25y = 0$ .
- $y'' + 4y' + 13y = 0$ .

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Real roots

Complex roots

## The Vibrating Spring

Newton's second law:  $ma = \text{total force}$ .

- Forces acting:
  - $\diamond$  Gravity  $mg$ .
  - $\diamond$  Restoring force  $R(x)$ .
  - $\diamond$  Damping force  $D(v)$ .
  - $\diamond$  External force  $F(t)$ .

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- Newton's law becomes

$$ma = mg + R(x) + D(v) + F(t)$$

- Hooke's law:  $R(x) = -kx$ .  $k > 0$  is the *spring constant*.

- Spring-mass equilibrium  $x_0 = mg/k$ .

- Set  $y = x - x_0$ . Newton's law becomes

$$my'' = -ky + D(y') + F(t).$$

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- Damping force  $D(y') = -\mu y'$ .

- Newton's law becomes

$$my'' = -ky - \mu y' + F(t), \quad \text{or}$$

$$my'' + \mu y' + ky = F(t), \quad \text{or}$$

$$y'' + \frac{\mu}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t).$$

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Vibrating spring

### RLC Circuit



$$LI'' + RI' + \frac{1}{C}I = E'(t), \quad \text{or}$$

$$I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t).$$

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Vibrating spring equation

## Harmonic Motion

- Spring:  $y'' + \frac{\mu}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t)$ .
- Circuit:  $I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t)$ .
- Essentially the same equation. Use

$$x'' + 2cx' + \omega_0^2x = f(t).$$

- The equation for *harmonic motion*.

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$$x'' + 2cx' + \omega_0^2x = f(t).$$

- $\omega_0$  is the *natural frequency*.
  - ◊ Spring:  $\omega_0 = \sqrt{k/m}$ .
  - ◊ Circuit:  $\omega_0 = \sqrt{1/LC}$ .
- $c$  is the *damping constant*.
- $f(t)$  is the *forcing term*.

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## Simple Harmonic Motion

No forcing, and no damping.

$$x'' + \omega_0^2x = 0$$

- $p(\lambda) = \lambda^2 + \omega_0^2$ ,  $\lambda = \pm i\omega_0$ .

Fundamental set of solutions

$$x_1(t) = \cos \omega_0 t \quad \& \quad x_2(t) = \sin \omega_0 t.$$

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Harmonic motion

General solution

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.$$

- Every solution is periodic with frequency  $\omega_0$ .
  - ◊  $\omega_0$  is the natural frequency.
  - ◊ The period is  $T = 2\pi/\omega_0$ .

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### Amplitude and Phase

- Put  $C_1$  and  $C_2$  in polar coordinates:

$$C_1 = A \cos \phi, \text{ \& } C_2 = A \sin \phi.$$

- Then  $x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$

$$= A \cos(\omega_0 t - \phi).$$

- $A$  is the *amplitude*;  $A = \sqrt{C_1^2 + C_2^2}$ .
- $\phi$  is the *phase*;  $\tan \phi = C_2/C_1$ .

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### Examples

- $C_1 = 3, C_2 = 4 \Rightarrow A = 5, \phi = 0.9273$ .
- $C_1 = -3, C_2 = 4 \Rightarrow A = 5, \phi = 2.2143$ .
- $C_1 = -3, C_2 = -4 \Rightarrow A = 5, \phi = -2.2143$ .

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Amplitude & phase

### Example

$$x'' + 16x = 0, x(0) = -2 \text{ \& } x'(0) = 4$$

- Natural frequency:  $\omega_0^2 = 16 \Rightarrow \omega_0 = 4$ .
- General solution:  $x(t) = C_1 \cos 4t + C_2 \sin 4t$ .
- IC:  $-2 = x(0) = C_1$ , and  $4 = x'(0) = 4C_2$ .
- Solution

$$\begin{aligned} x(t) &= -2 \cos 2t + \sin 2t \\ &= \sqrt{5} \cos(2t - 2.6779). \end{aligned}$$

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Amplitude &amp; phase

### Damped Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = 0$$

- $p(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$ ; roots  $-c \pm \sqrt{c^2 - \omega_0^2}$ .
- Three cases
  - ◊  $c < \omega_0$  *Underdamped*
  - ◊  $c > \omega_0$  *Overdamped*
  - ◊  $c = \omega_0$  *Critically damped*

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Harmonic motion

### Underdamped

- $c < \omega_0$
- Two complex roots  $\lambda$  and  $\bar{\lambda}$ , where  $\lambda = -c + i\omega$  and  $\omega = \sqrt{\omega_0^2 - c^2}$ .
- General solution

$$\begin{aligned} x(t) &= e^{-ct} [C_1 \cos \omega t + C_2 \sin \omega t] \\ &= A e^{-ct} \cos(\omega t - \phi) \end{aligned}$$

Amplitude &amp; phase

### Overdamped

- $c > \omega_0$ , so two real roots

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2}$$

$$\lambda_2 = -c + \sqrt{c^2 - \omega_0^2}.$$

- $\lambda_1 < \lambda_2 < 0$ .
- General solution

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

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### Critically Damped

- $c = \omega_0$
- One negative real root  $\lambda = -c$  with multiplicity 2.
- General solution

$$x(t) = e^{-ct}[C_1 + C_2 t].$$

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