Math 211

Lecture #33

Harmonic Motion

Inhomogeneous Equations

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Harmonic Motion

- Spring: $y'' + \frac{\mu}{m} y' + \frac{k}{m} y = \frac{1}{m} F(t)$.
- Circuit: $I'' + \frac{R}{L} I' + \frac{1}{LC} I = \frac{1}{L} E'(t)$.
- Essentially the same equation. Use

$$x'' + 2cx' + \omega_0^2 x = f(t).$$

- The equation for *harmonic motion.*

$$x'' + 2cx' + \omega_0^2 x = f(t).$$

- ω_0 is the *natural frequency.*
  - Spring: $\omega_0 = \sqrt{k/m}$.
  - Circuit: $\omega_0 = \sqrt{1/LC}$.
- c is the *damping constant.*
- $f(t)$ is the *forcing term.*

Simple Harmonic Motion

- No forcing , and no damping:

$$x'' + \omega_0^2 x = 0$$

- $p(\lambda) = \lambda^2 + \omega_0^2$, $\lambda = \pm i\omega_0$.
- Fundamental set of solutions

$$x_1(t) = \cos \omega_0 t \quad \& \quad x_2(t) = \sin \omega_0 t.$$

- General solution

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.$$

  - Every solution is periodic with frequency ω_0.
  - ω_0 is the natural frequency.
  - The period is $T = 2\pi/\omega_0$.

Damped Harmonic Motion

$$x'' + 2cx' + \omega_0^2 x = 0$$

- $p(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2$; roots $-c \pm \sqrt{c^2 - \omega_0^2}$.
- Three cases
  - ◇ $c < \omega_0$ *Underdamped*
  - ◇ $c > \omega_0$ *Overdamped*
  - ◇ $c = \omega_0$ *Critically damped*

Underdamped

- $c < \omega_0$
- Two complex roots λ and $\overline{\lambda}$, where $\lambda = -c + i\omega$ and $\omega = \sqrt{\omega_0^2 - c^2}$.
- General solution

$$\begin{aligned} x(t) &= e^{-ct}[C_1 \cos \omega t + C_2 \sin \omega t] \\ &= Ae^{-ct} \cos(\omega t - \phi) \end{aligned}.$$

Overdamped

- $c > \omega_0$, so two real roots

$$\lambda_1 = -c - \sqrt{c^2 - \omega_0^2}$$
$$\lambda_2 = -c + \sqrt{c^2 - \omega_0^2}.$$

- $\lambda_1 < \lambda_2 < 0.$
- General solution

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

Critically Damped

- $c = \omega_0$
- One negative real root $\lambda = -c$ with multiplicity 2.
- General solution

$$x(t) = e^{-ct}[C_1 + C_2 t].$$

Inhomogeneous Equations

$$y'' + py' + qy = f(t)$$

- Corresponding homogeneous equation

$$y'' + py' + qy = 0$$

  - ◇ We know how to find a fundamental set of solutions y_1 and y_2.
  - ◇ The general solution of the homogeneous equation is $y_h(t) = C_1 y_1(t) + C_2 y_2(t)$.

Theorem: Assume

- $y_p(t)$ is a particular solution to the inhomogeneous equation $y'' + py' + qy = f(t)$;
- $y_1(t)$ & $y_2(t)$ is a fundamental set of solutions to the homogeneous equation $y'' + py' + qy = 0$.

Then the general solution to the inhomogeneous equation is

$$y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t).$$

Method of Undetermined Coefficients

$$y'' + py' + qy = f(t)$$

- If the forcing term $f(t)$ has a form which is replicated under differentiation, then look for a particular solution of the same general form as the forcing term.

Exponential Forcing Term

$$y'' + py' + qy = Ce^{at}$$

- Example: $y'' + 3y' + 2y = 4e^{-3t}$
- Try $y_p(t) = ae^{-3t}$; a to be determined.

$$y_p'' + 3y_p' + 2y_p = 2ae^{-3t}$$

- Particular solution if $2a = 4$, or $a = 2$.

$$y_p(t) = 2e^{-3t}.$$

- Homogeneous equation:

$$y'' + 3y' + 2y = 0 \quad \text{ODE}$$
$$\lambda^2 + 3\lambda + 2 = 0 \quad \text{Ch. poly.}$$
$$(\lambda + 2)(\lambda + 1) = 0$$

- Fund. set of sol'ns: e^{-2t} & e^{-t}.
- General solution to the inhomogeneous equation is

$$y(t) = 2e^{-3t} + C_1 e^{-t} + C_2 e^{-2t}.$$

Trigonometric Forcing Term

$$y'' + py' + qy = A\cos\omega t + B\sin\omega t$$

- Example: $y'' + 4y' + 5y = 4\cos 2t - 3\sin 2t$
- Try $y_p(t) = a\cos 2t + b\sin 2t$

$$y_p'' + 4y_p' + 5y_p = (a+8b)\cos 2t + (b-8a)\sin 2t.$$

- Particular solution if $a + 8b = 4,\ b - 8a = -3.$
  $\Leftrightarrow a = 28/65$ and $b = 29/65.$

- $y_p(t) = [28\cos 2t + 29\sin 2t]/65.$
- Homogeneous equation: $y'' + 4y' + 5y = 0$
  - Characteristic polynomial: $\lambda^2 + 4\lambda + 5 = 0$
  - Roots: $\lambda = -2 \pm i$
- Fund. set of sol'ns: $e^{-2t}\cos t$ & $e^{-2t}\sin t$.
- General solution to the inhomogeneous equation:

$$y(t) = [28\cos 2t + 29\sin 2t]/65 \\ + e^{-2t}[C_1\cos t + C_2\sin t].$$

Complex Method

$$x'' + px' + qx = A\cos\omega t \quad \text{or}$$

$$y'' + py' + qy = A\sin\omega t.$$

- Solve $z'' + pz' + qz = Ae^{i\omega t}$.
- $x_p(t) = \mathrm{Re}(z(t))$ and $y_p(t) = \mathrm{Im}(z(t))$.

- Example: $x'' + 4x' + 5x = 4\cos 2t$
- Solve $z'' + 4z' + 5z = 4e^{2it}$.
- Try $z(t) = ae^{2it}$.

$$z'' + 4z' + 5z = (1 + 8i)ae^{2it}$$

- Particular solution if $(1 + 8i)a = 4$ or

$$a = \frac{4}{1 + 8i} = \frac{4(1 - 8i)}{1 + 64} = \frac{4 - 32i}{65}.$$

Return

- Particular solution

$$
\begin{aligned}
z(t) &= (4 - 32i)e^{2it}/65 \\
&= (4 - 32i)[\cos 2t + i \sin 2t]/65 \\
&= [4\cos 2t + 32 \sin 2t]/65 \\
&\quad + i[4 \sin 2t - 32 \cos 2t]/65.
\end{aligned}
$$

$$
\begin{aligned}
x_p(t) &= \mathrm{Re}(z(t)) \\
&= [4\cos 2t + 32\sin 2t]/65.
\end{aligned}
$$

Polynomial Forcing Term

$$y'' + py' + qy = P(t)$$

- Example: $y'' - 3y' + 2y = 1 - 4t.$
- Try $y(t) = a + bt.$

$$y'' - 3y' + 2y = (a - 3b) + 2bt.$$

- Particular solution if

$$\begin{aligned} a - 3b &= 1 \\ 2b &= -4 \end{aligned} \quad \text{or} \quad \begin{aligned} b &= -2 \\ a &= -5 \end{aligned}$$

- Particular solution

$$y(t) = -5 - 2t.$$

- General solution

$$y(t) = -5 - 2t + C_1 e^t + C_2 e^{2t}.$$

Return

Exceptional Cases

- Example: $y'' - 3y' + 2y = 3e^t$.
- Try $y(t) = ae^t$

$$y'' - 3y' + 2y = 0.$$

- The method does not work because e^t is a solution to the associated homogeneous equation.

- Try $y(t) = ate^t$

$$y'' - 3y' + 2y = -ae^t$$

- Particular solution if $a = -3$.
- General solution

$$y(t) = -3te^t + C_1e^t + C_2e^{2t}.$$

- If the suggested solution does not work, multiply it by t and try again.

Combination Forcing Term

Example $y'' + 5y' + 6y = 2e^{2t} - 5\cos t$

- Solve

$$y_1'' + 5y_1' + 6y_1 = 2e^{2t}$$

$$y_2'' + 5y_2' + 6y_2 = -5\cos t$$

- Set $y(t) = y_1(t) + y_2(t)$.