Math 211

Lecture #34

Inhomgeneous Equations
Forced Harmonic Motion

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Inhomogeneous Equations

Theorem: Assume

- $y_p(t)$ is a particular solution to the inhomogeneous equation $y'' + py' + qy = f(t)$;
- $y_1(t)$ & $y_2(t)$ is a fundamental set of solutions to the homogeneous equation $y'' + py' + qy = 0$.

Then the general solution to the inhomogeneous equation is

$$y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t).$$

Method of Undetermined Coefficients

$$y'' + py' + qy = f(t)$$

- If the forcing term $f(t)$ has a form which is replicated under differentiation, then look for a particular solution of the same general form as the forcing term.

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Examples

- Exponential Forcing Term: $y'' + py' + qy = Ce^{at}$
  - ◇ Try $y_p(t) = Ae^{ct}$; A to be determined.
- Trigonometric Forcing Term:
  - ◇ $y'' + py' + qy = A\cos\omega t + B\sin\omega t$
    - ⋆ Try $y_p(t) = a\cos\omega t + b\sin\omega t$
  - ◇ $x'' + px' + qx = A\cos\omega t$ or

    $y'' + py' + qy = A\sin\omega t.$
    - ⋆ Solve $z'' + pz' + qz = Ae^{i\omega t}$.
    - ⋆ $x_p(t) = \mathrm{Re}(z(t))$ and $y_p(t) = \mathrm{Im}(z(t))$.

Polynomial Forcing Term

$$y'' + py' + qy = P(t)$$

- Try $y_p(t) = Q(t)$, a polynomial of the same degree.
- Example: $y'' - 3y' + 2y = 1 - 4t.$
  - ◇ Particular solution: $y_p(t) = -5 - 2t.$
  - ◇ General solution

$$y(t) = -5 - 2t + C_1 e^t + C_2 e^{2t}.$$

Exceptional Cases

- Example: $y'' - 3y' + 2y = 3e^t$.
- Try $y_p(t) = ae^t$

$$y_p'' - 3y_p' + 2y_p = 0.$$

- The method does not work because e^t is a solution to the associated homogeneous equation.

- Try $y_p(t) = ate^t$

$$y_p'' - 3y_p' + 2y_p = -ae^t$$

  - Particular solution if $a = -3$.
  - General solution

$$y(t) = -3te^t + C_1e^t + C_2e^{2t}.$$

- If the suggested solution does not work, multiply it by t and try again.

Combination Forcing Term

Example $y'' + 5y' + 6y = 2e^{2t} - 5\cos t$

- Solve

$$y_1'' + 5y_1' + 6y_1 = 2e^{2t}$$

$$y_2'' + 5y_2' + 6y_2 = -5\cos t$$

- Set $y(t) = y_1(t) + y_2(t)$.

Forced Harmonic Motion

Assume an oscillatory forcing term:

$$y'' + 2cy' + \omega_0^2 y = A \cos \omega t$$

- A is the forcing amplitude
- ω is the forcing frequency
- ω_0 is the natural frequency.
- c is the damping constant.

Forced Undamped Motion

$$y'' + \omega_0^2 y = A\cos\omega t$$

- Homogeneous equation

$$y'' + \omega_0^2 y = 0$$

  - ◇ General solution

$$y(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t.$$

  - ◇ If $\omega = \omega_0$ we have an exceptional case.

- $\omega \neq \omega_0$

$$y'' + \omega_0^2 y = A\cos\omega t$$

◇ Look for a particular solution of the form

$$x_p(t) = a\cos\omega t + b\sin\omega t.$$

◇ We find

$$x_p(t) = \frac{A}{\omega_0^2 - \omega^2}\cos\omega t.$$

- $\omega \neq \omega_0$
  - General solution

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{A}{\omega_0^2 - \omega^2} \cos \omega t.$$

  - Initial conditions $x(0) = x'(0) = 0 \;\Rightarrow$

$$x(t) = \frac{A}{\omega_0^2 - \omega^2} [\cos \omega t - \cos \omega_0 t].$$

  - Example: $\omega_0 = 9$, $\omega = 8$, $A = \omega_0^2 - \omega^2 = 17$.

$$x(t) = \cos 9t - \cos 8t.$$

- $\omega \neq \omega_0$
  - ◇ Set

$$\overline{\omega} = \frac{\omega_0 + \omega}{2} \quad \text{and} \quad \delta = \frac{\omega_0 - \omega}{2}.$$

$$\Rightarrow \omega = \overline{\omega} - \delta \quad \text{and} \quad \omega_0 = \overline{\omega} + \delta, \quad \text{and}$$

$$x(t) = \frac{A}{\omega_0^2 - \omega^2}[\cos \omega t - \cos \omega_0 t]$$

$$= \frac{A \sin \delta t}{2\overline{\omega}\delta} \sin \overline{\omega} t.$$

- $\omega \neq \omega_0$
  - Example:

$$\overline{\omega} = 8.5 \quad \text{and} \quad \delta = 0.5.$$

  - Envelope: Slow oscillation with frequency δ.
  - Fast oscillation with frequency $\overline{\omega}$ and varying amplitude.
  - Beats.

- $\omega = \omega_0$

$$y'' + \omega_0^2 y = A\cos\omega_0 t.$$

◇ We have an exceptional case. Try

$$x_p(t) = t[a\cos\omega t + b\sin\omega t].$$

◇ We find

$$x_p(t) = \frac{A}{2\omega_0} t\sin\omega_0 t.$$

◇ General solution

$$x(t) = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{A}{2\omega_0} t\sin\omega_0 t.$$

- $\omega = \omega_0$
  - ◇ Initial conditions $x(0) = x'(0) = 0 \Rightarrow$

$$x(t) = \frac{A}{2\omega_0} t \sin \omega_0 t.$$

    - ⋆ Example: $\omega_0 = 5$, and $A = 2\omega_0 = 10$.

$$x(t) = t \sin 5t.$$

  - ◇ Oscillation with increasing amplitude.
  - ◇ First example of *resonance.*
    - ⋆ Driving at the natural frequency can cause oscillations that grow out of control.

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