# Math 211

Lecture #36

Nonlinear Systems

April 18, 2001

# Interacting Species

- Two species with populations $x_1$ & $x_2$.
- Interaction between the species can be helpful or detrimental.
- Model

$$x_1' = r_1 x_1$$

$$x_2' = r_2 x_2$$

- $r_1$ & $r_2$ are the *reproductive rates.*

# Reproductive Rates

- If $x_2 = 0$ the reproductive rate for $x_1$ is

$$r_1 = a_1 - b_1 x_1.$$

  - ◇ $a_1 > 0 \Rightarrow$ natural growth.
  - ◇ $a_1 < 0 \Rightarrow$ natural decline.
  - ◇ $b_1 = 0$ Malthusian growth.
  - ◇ $b_1 > 0$ logistic growth.

- If $x_2 > 0$ the reproductive rate for $x_1$ is

$$r_1 = a_1 - b_1 x_1 + c_1 x_2.$$

  - ◇ $c_1 > 0 \Rightarrow$ interaction is helpful to $x_1$.
  - ◇ $c_1 < 0 \Rightarrow$ interaction is detrimental to $x_1$.

- The reproduction rate for $x_2$ is

$$r_2 = a_2 - b_2 x_2 + c_2 x_1.$$

# Interacting Species Model

$$x_1' = (a_1 - b_1 x_1 + c_1 x_2) x_1$$

$$x_2' = (a_2 - b_2 x_2 + c_2 x_1) x_2$$

# Predator Prey Model

Rabbits & foxes, fish & sharks, and cottony cushion scale insect & ladybird beetle.

- $F$ = fish & $S$ = sharks.

$$F' = (a - bS)F$$

$$S' = (-c + dF)S$$

or

$$F' = (a - eF - bS)F$$

$$S' = (-c + dF)S$$

# Example of Predator Prey

$$F' = (3 - 3S)F$$

$$S' = (-1 + 3F)S$$

or

$$F' = (3 - 3F - 3S)F$$

$$S' = (-1 + 3F)S$$

Return

# Competing Species

Cattle and sheep.

- $x_1$ and $x_2$ competing for resources.

$$x_1' = (a_1 - b_1 x_1 + c_1 x_2) x_1$$

$$x_2' = (a_2 - b_2 x_2 + c_2 x_1) x_2$$

- $a_i > 0$ , $b_i > 0$, & $c_i < 0$

# Example of Competing Species

$$x' = (5 - 2x - y)x$$

$$y' = (7 - 2x - 3y)y$$

# Linearization

Principal idea of differential calculus:

- Approximate nonlinear mathematical objects by linear ones.
- Approximate $f(y) = f(y_0 + h)$ near $y_0$ by the linear function $L(h) = f(y_0) + f'(y_0)h$.

$$f(y_0 + h) = f(y_0) + f'(y_0)h + R(h)$$

$$\text{where } \lim_{h \to 0} \frac{R(h)}{h} = 0.$$

# Linearization of an ODE

$$y' = f(y)$$

- Assume $f(y_0) = 0$ and $f'(y_0) \neq 0$.
- Set $y = y_0 + u$. Get

$$u' = f'(y_0)u + R(u)$$

- Approximate by the linear equation

$$\tilde{u}' = f'(y_0)\tilde{u}$$

- If $f'(y_0) \neq 0$ the equilibrium point of the linearization at $0$ has the same stability properties as that of the nonlinear equation at $y_0$.
  - ◇ $f'(y_0) > 0 \Rightarrow y_0$ is unstable.
  - ◇ $f'(y_0) < 0 \Rightarrow y_0$ is asymptotically stable.
- We can solve the linearization explicitly.

# Linearization of a Planar System

$$x' = f(x, y)$$
$$y' = g(x, y)$$

- $(x_0, y_0)$ is an equilibrium point so

$$f(x_0, y_0) = g(x_0, y_0) = 0$$

We have by Taylor's theorem

$$f(x_0 + u, y_0 + v)$$

$$= \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v)$$

$$g(x_0 + u, y_0 + v)$$

$$= \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v)$$

where $\dfrac{R_f(u, v)}{\sqrt{u^2 + v^2}} \to 0$ and $\dfrac{R_g(u, v)}{\sqrt{u^2 + v^2}} \to 0$

- Set $x = x_0 + u$ and $y = y_0 + v$. The system becomes

$$u' = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v)$$

$$v' = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v)$$

# Linearization at $(x_0, y_0)$

$$\tilde{u}' = \frac{\partial f}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0)\tilde{v}$$

$$\tilde{v}' = \frac{\partial g}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0)\tilde{v}$$

- This is a linear system.
  - ◇ We can solve it explicitly.
  - ◇ Does it give information about the original system?

# Matrix Form of the Linearization

Set $\mathbf{u} = (\tilde{u}, \tilde{v})^T$ and introduce the *Jacobian matrix*

$$J = \begin{pmatrix} \dfrac{\partial f}{\partial x}(x_0, y_0) & \dfrac{\partial f}{\partial y}(x_0, y_0) \\ \dfrac{\partial g}{\partial x}(x_0, y_0) & \dfrac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

- The linearization becomes

$$\mathbf{u}' = J\mathbf{u}.$$

**Theorem:** Consider the planar system

$$x' = f(x, y)$$
$$y' = g(x, y)$$

where $f$ and $g$ are continuously differentiable. Suppose that $(x_0, y_0)$ is an equilibrium point. If the linearization at $(x_0, y_0)$ has a generic equilibrium point at the origin, then the equilibrium point at $(x_0, y_0)$ is of the same type.

# Generic Equilibrium Points

- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - All occupy large open subsets of the trace-determinant plane.
- Nongeneric types
  - Center and eight others. Occupy pieces of the boundaries.

# Examples

- Predator prey
- Competing species
- Center

$$x' = y + \alpha x(x^2 + y^2)$$

$$y' = -x + \alpha y(x^2 + y^2)$$