Math 211

Lecture #37

Linearization

April 20, 2001

Linearization of a Planar System

$$x' = f(x, y)$$
$$y' = g(x, y)$$

- (x_0, y_0) is an equilibrium point so

$$f(x_0, y_0) = g(x_0, y_0) = 0$$

We have by Taylor's theorem

$$f(x_0 + u, y_0 + v) = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v)$$

$$g(x_0 + u, y_0 + v) = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v)$$

where $\frac{R_f(u,v)}{\sqrt{u^2 + v^2}} \to 0$ and $\frac{R_g(u,v)}{\sqrt{u^2 + v^2}} \to 0$

- Set $x = x_0 + u$ and $y = y_0 + v$. The system becomes

$$u' = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v)$$

$$v' = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v)$$

Linearization at (x_0, y_0)

$$\tilde{u}' = \frac{\partial f}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0)\tilde{v}$$
$$\tilde{v}' = \frac{\partial g}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0)\tilde{v}$$

- This is a linear system.
  - We can solve it explicitly.
  - Does it give information about the original system?

Matrix Form of the Linearization

Set $\mathbf{u} = (\tilde{u}, \tilde{v})^T$ and introduce the *Jacobian matrix*

$$J = \begin{pmatrix} \dfrac{\partial f}{\partial x}(x_0, y_0) & \dfrac{\partial f}{\partial y}(x_0, y_0) \\ \dfrac{\partial g}{\partial x}(x_0, y_0) & \dfrac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

- The linearization becomes

$$\mathbf{u}' = J\mathbf{u}.$$

Theorem: Consider the planar system

$$x' = f(x, y)$$

$$y' = g(x, y)$$

where f and g are continuously differentiable. Suppose that (x_0, y_0) is an equilibrium point. If the linearization at (x_0, y_0) has a generic equilibrium point at the origin, then the equilibrium point at (x_0, y_0) is of the same type.

Generic Equilibrium Points

- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - ◇ All occupy large open subsets of the trace-determinant plane.
- Nongeneric types
  - ◇ Center and eight others. Occupy pieces of the boundaries.

Examples

- Predator prey

$$F' = (3 - 3S)F$$

$$S' = (-1 + 3F)S$$

or

$$F' = (3 - 3F - 3S)F$$

$$S' = (-1 + 3F)S$$

- Competing species

$$x' = (5 - 2x - y)x$$
$$y' = (7 - 2x - 3y)y$$

- Center

$$x' = y + \alpha x(x^2 + y^2)$$
$$y' = -x + \alpha y(x^2 + y^2)$$

  - ◇ $\alpha > 0 \Rightarrow (0,0)^T$ is unstable.
  - ◇ $\alpha < 0 \Rightarrow (0,0)^T$ is a sink.

Higher Dimensional Systems

Autonomous equation $\mathbf{y}' = \mathbf{f}(\mathbf{y})$

- $\mathbf{y} = (y_1, y_2, \cdots, y_n)^T$
- $\mathbf{f}(\mathbf{y}) = (f_1(\mathbf{y}), f_2(\mathbf{y}), \cdots, f_n(\mathbf{y}))^T$
- J is the Jacobian matrix

$$\mathbf{f}(\mathbf{y}_0 + \mathbf{u}) = J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u})$$

$$\text{where } \lim_{\mathbf{u}\to\mathbf{0}} \frac{\mathbf{R}(\mathbf{u})}{|\mathbf{u}|} = \mathbf{0}.$$

Return

Set $\mathbf{y} = \mathbf{y}_0 + \mathbf{u}$. The system becomes

$$\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u} + \mathbf{R}(\mathbf{u}).$$

The linearization is

$$\mathbf{u}' = J(\mathbf{y}_0)\mathbf{u}.$$

Theorem: Suppose that $\mathbf{y}_0$ is an equilibrium point for $\mathbf{y}' = \mathbf{f}(\mathbf{y})$. Let J be the Jacobian of $\mathbf{f}$ at $\mathbf{y}_0$.

1. Suppose that the real part of every eigenvalue of J is negative. Then $\mathbf{y}_0$ is an asymptotically stable equilibrium point.

2. Suppose that J has at least one eigenvalue with positive real part. Then $\mathbf{y}_0$ is an unstable equilibrium point.

Example

$$x' = -2x - 4y + 2xy$$

$$y' = x - 6y + x^2 - y^2$$

- One eigenvalue $\lambda = -4$ of algebraic multiplicity 2.
- First theorem does not apply.
- Second theorem does apply. The origin is a sink.

The Lorenz System

$$x' = -ax + ay$$
$$y' = rx - y - xz$$
$$z' = -bz + xy$$

Equilibrium points.

- $(r \leq 1)$ $(0, 0, 0)$
- $(r > 1)$ Set $s = \sqrt{b(r-1)}$.

  $(0, 0, 0)$, $\mathbf{c}^+ = (s, s, r-1)$ & $\mathbf{c}^- = (-s, -s, r-1)$

- The Jacobian is

$$J = \begin{pmatrix} -a & a & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}$$

  - ◇ Use $a = 10$ and $b = 8/3$.
  - ◇ $(0, 0, 0)$
    - ⋆ If $r < 1$ $(0, 0, 0)$ is asymptotically stable.
    - ⋆ If $r > 1$ $(0, 0, 0)$ is unstable.

◇ $\mathbf{c}^+$ and $\mathbf{c}^-$

- ⋆ For $1 < r < 470/19 \approx 24.74$, $\mathbf{c}^+$ and $\mathbf{c}^-$ are asymptotically stable.
- ⋆ For $r > 470/19 \approx 24.74$, $\mathbf{c}^+$ and $\mathbf{c}^-$ are unstable.

- As r varies the Lorenz system displays a wide variety of behaviors.
  - For $r = 28$ we have Lorenz's strange attractor.
  - For $r = 100$ there is a periodic attractor.
  - For $r = 200$ there is another strange attractor.