

Math 211

Lecture #39

Limit Sets

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Limit Sets

Definition: The (*forward*) *limit set* of the solution $\mathbf{y}(t)$ that starts at \mathbf{y}_0 is the set of all limit points of the solution curve. It is denoted by $\omega(\mathbf{y}_0)$.

- $\mathbf{x} \in \omega(\mathbf{y}_0)$ if there is a sequence $t_k \rightarrow \infty$ such that $\mathbf{y}(t_k) \rightarrow \mathbf{x}$.
- What kinds of sets can be limit sets?
 - ◊ Equilibrium points.
 - ◊ Periodic orbits.

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Properties of Limit Sets

Theorem: Suppose that the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ is defined in the set U .

1. If the solution curve starting at \mathbf{y}_0 stays in a bounded subset of U , then the limit set $\omega(\mathbf{y}_0)$ is not empty.
2. Any limit set is both positively and negatively invariant.

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Example

$$x' = -y + x(1 - x^2 - y^2)$$

$$y' = x + y(1 - x^2 - y^2)$$

- In polar coordinates this is

$$r' = r(1 - r^2)$$

$$\theta' = 1$$

- Solution curves approach the unit circle.

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[Definition](#)

Limit Cycle

Definition: A *limit cycle* is a closed solution curve which is the limit set of nearby solution curves. If the solution curves spiral into the limit cycle as $t \rightarrow \infty$, it is a *attracting limit cycle*. If they spiral into the limit cycle as $t \rightarrow -\infty$, it is a *repelling limit cycle*.

- In the example the unit circle is a limit cycle.

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Types of Limit Set

- A limit cycle is a new type of phenomenon.
- However, the limit set is a periodic orbit, so the type of limit set is not new.
- We still have only two types.

Example

$$x' = (y + x/5)(1 - x^2)$$

$$y' = -x(1 - y^2)$$

- The limit set of any solution that starts in the unit square is the boundary of the unit square.

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Planar Graph

Definition: A *planar graph* is a collection of points, called *vertices*, and non-intersecting curves, called *edges*, which connect the vertices. If the edges each have a direction the graph is said to be *directed*.

- The boundary of the unit square in the example is a directed planar graph.

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Theorem: If S is a limit set of a solution of a planar system defined in a set $U \subset \mathbf{R}^2$, then S is one of the following:

- An equilibrium point
- A closed solution curve
- A directed planar graph with vertices that are equilibrium points, and edges which are solution curves.

These are called the *Poincaré-Bendixson alternatives*.

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Remarks

- These are the only possibilities.
- The closed solution curve could be a limit cycle.
- If a vertex of a limiting planar graph is a generic equilibrium point, then it must be a saddle point. The edges connecting this point must be separatrices.

[Return](#)[Poincaré-Bendixson alternatives](#)

Poincaré-Bendixson Theorem

Theorem: Suppose that R is a closed and bounded planar region that is positively invariant for a planar system. If R contains no equilibrium points, then there is a closed solution curve in R .

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