Math 211

Lecture #39

Limit Sets

April 25, 2001

Limit Sets

Definition: The *(forward) limit set* of the solution $\mathbf{y}(t)$ that starts at $\mathbf{y}_0$ is the set of all limit points of the solution curve. It is denoted by $\omega(\mathbf{y}_0)$.

- $\mathbf{x} \in \omega(\mathbf{y}_0)$ if there is a sequence $t_k \to \infty$ such that $\mathbf{y}(t_k) \to \mathbf{x}$.
- What kinds of sets can be limit sets?
  - Equilibrium points.
  - Periodic orbits.

Properties of Limit Sets

Theorem: Suppose that the system $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ is defined in the set U.

1. If the solution curve starting at $\mathbf{y}_0$ stays in a bounded subset of U, then the limit set $\omega(\mathbf{y}_0)$ is not empty.

2. Any limit set is both positively and negatively invariant.

Return

Example

$$x' = -y + x(1 - x^2 - y^2)$$
$$y' = x + y(1 - x^2 - y^2)$$

- In polar coordinates this is

$$r' = r(1 - r^2)$$
$$\theta' = 1$$

- Solution curves approach the unit circle.

Limit Cycle

Definition: A *limit cycle* is a closed solution curve which is the limit set of nearby solution curves. If the solution curves spiral into the limit cycle as $t \longrightarrow \infty$, it is a *attracting limit cycle*. If they spiral into the limit cycle as $t \longrightarrow -\infty$, it is a *repelling limit cycle*.

- In the example the unit circle is a limit cycle.

Return

Types of Limit Set

- A limit cycle is a new type of phenomenon.
- However, the limit set is a periodic orbit, so the type of limit set is not new.
- We still have only two types.

Example

$$x' = (y + x/5)(1 - x^2)$$

$$y' = -x(1 - y^2)$$

- The limit set of any solution that starts in the unit square is the boundary of the unit square.

Planar Graph

Definition: A *planar graph* is a collection of points, called *vertices*, and non-intersecting curves, called *edges*, which connect the vertices. If the edges each have a direction the graph is said to be *directed*.

- The boundary of the unit square in the example is a directed planar graph.

Return

Theorem: If S is a limit set of a solution of a planar system defined in a set $U \subset \mathbf{R}^2$, then S is one of the following:

- An equilibrium point
- A closed solution curve
- A directed planar graph with vertices that are equilibrium points, and edges which are solution curves.

These are called the *Poincaré-Bendixson alternatives.*

Return

Remarks

- These are the only possibilities.
- The closed solution curve could be a limit cycle.
- If a vertex of a limiting planar graph is a generic equilibrium point, then it must be a saddle point. The edges connecting this point must be separatrices.

Poincaré-Bendixson Theorem

Theorem: Suppose that R is a closed and bounded planar region that is positively invariant for a planar system. If R contains no equilibrium points, then there is a closed solution curve in R.