Some topics related to p-adic numbers, absolute value functions on fields, and ultrametrics

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Preface

The usual absolute values of real and complex numbers may be considered as examples of a broader notion of an absolute value function on a field. In fact, one can start with the standard absolute value function on the rationals, and the real line $\mathbf{R}$ may be considered as a completion of the field $\mathbf{Q}$ of rational numbers with respect to the metric associated to the standard absolute value function.

Similarly, if p is a prime number, then the *p-adic absolute value* $|x|_p$ of a rational number x may be defined in a standard way. The field $\mathbf{Q}_p$ of *p-adic numbers* may be obtained by completing $\mathbf{Q}$ with respect to the corresponding *p-adic metric* $|x - y|_p$.

The p-adic absolute value satisfies a stronger version of the triangle inequality, to wit,

$$|x + y|_p \le \max(|x|_p, |x|_p). \tag{0.0.1}$$

This means that the p-adic metric is an *ultrametric*. Metrics and ultrametrics are discussed in Section 1.1, and absolute value functions on arbitrary fields are discussed in Section 1.2.

Some basic references concerning absolute value functions on fields and p-adic numbers in particular include [33, 70], Chapter V of [95], and Chapter XII of [111].

Contents

Chapter 1

Metrics and absolute value functions

1.1 Metrics, ultrametrics, and quasimetrics

Let X be a set, and let $d(x, y)$ be a nonnegative real-valued function defined for $x, y \in X$. We say that $d(\cdot, \cdot)$ is a *metric* if it satisfies the following three conditions, as usual. First,

$$d(x, y) = 0 \text{ if and only if } x = y. \tag{1.1.1}$$

Second,

$$d(x, y) = d(y, x) \tag{1.1.2}$$

for every $x, y \in X$. Third,

$$d(x, z) \le d(x, y) + d(y, z) \tag{1.1.3}$$

for every $x, y, z \in X$, which is known as the *triangle inequality*.

If $d(\cdot, \cdot)$ satisfies (1.1.1), (1.1.2), and

$$d(x, z) \le \max(d(x, y), d(y, z)) \tag{1.1.4}$$

for every $x, y, z \in X$, then $d(\cdot, \cdot)$ is said to be an *ultrametric* on X. Note that (1.1.4) implies (1.1.3), so that an ultrametric is a metric in particular.

The *discrete metric* is defined on X by putting $d(x, y)$ equal to 1 when $x \ne y$, and to 0 when $x = y$. One can check that this is an ultrametric on X. The standard Euclidean metric on the real line $\mathbf{R}$ is not an ultrametric.

Let us say that $d(\cdot, \cdot)$ is a *quasimetric* on X if it satisfies (1.1.1), (1.1.2), and

$$d(x, z) \le C\,(d(x, y) + d(y, z)) \tag{1.1.5}$$

for some nonnegative real number C and all $x, y, z \in X$. Equivalently, this means that there is a nonnegative real number C_0 such that

$$d(x, z) \le C_0\, \max(d(x, y), d(y, z)). \tag{1.1.6}$$

More precisely, (1.1.6) implies (1.1.5), with $C = C_0$. Similarly, (1.1.5) implies (1.1.6), with

$$C_0 = 2\,C. \tag{1.1.7}$$

Of course, a metric on X is a quasimetric in particular.

Let a be a positive real number. If $d(\cdot,\cdot)$ satisfies (1.1.1) and (1.1.2), then

$$d(x,y)^a \tag{1.1.8}$$

has the same properties. If $d(\cdot,\cdot)$ is an ultrametric on X, then it is easy to see that

$$d(\cdot,\cdot)^a \text{ is an ultrametric on } X \tag{1.1.9}$$

too. Similarly, if $d(\cdot,\cdot)$ is a quasimetric on X that satisfies (1.1.6) for some nonnegative real number C_0, then (1.1.8) is a quasimetric on X, with

$$d(x,z)^a \le C_0^a \max(d(x,y)^a, d(y,z)^a) \tag{1.1.10}$$

for all $x, y, z \in X$. If $d(\cdot,\cdot)$ is the standard Euclidean metric on the real line and $a > 1$, then one can check that (1.1.8) is not a metric on $\mathbf{R}$.

1.2 Absolute value functions

Let k be a field, and let $|x|$ be a nonnegative real-valued function defined for $x \in k$. We say that $|\cdot|$ is an *absolute value function* on k if it satisfies the following three conditions. First, for each $x \in k$,

$$|x| = 0 \text{ if and only if } x = 0. \tag{1.2.1}$$

Second,

$$|x\,y| = |x|\,|y| \tag{1.2.2}$$

for all $x, y \in k$. Third,

$$|x + y| \le |x| + |y| \tag{1.2.3}$$

for all $x, y \in k$, which is another version of the *triangle inequality.*

If $|\cdot|$ satisfies (1.2.1), (1.2.2), and

$$|x + y| \le \max(|x|, |y|) \tag{1.2.4}$$

for every $x, y \in k$, then $|\cdot|$ is said to be an *ultrametric absolute value function* on k. This implies that $|\cdot|$ is an absolute value function on k in particular.

The *trivial absolute value function* is defined on k by putting $|x| = 1$ when $x \neq 0$, and of course $|0| = 0$. It is easy to see that this is an ultrametric absolute value function on k. The standard absolute value functions on $\mathbf{R}$ and the field $\mathbf{C}$ are absolute value functions in this sense, and not ultrametric absolute value functions.

This definition of an absolute value function corresponds to the definition of an *absolute value* in Definition 2.1.1 on p21 of [70], and on p283 of [111]. This also corresponds to a *real valuation* in Definition 1 on p211 of [95]. An ultrametric absolute value function is said to be *non-archimedean* in [70, 111], and the term *valuation* is also used in [111]. We shall use the term non-archimedean in a slightly different way here that turns out to be equivalent, essentially as in Definition 3 on p213 of [95]. This will be discussed further in Section 1.5.

Let p be a prime number. The *p-adic absolute value* $|x|_p$ of a rational number x is defined as follows. Of course, $|0|_p = 0$. If $x = p^j\,(a/b)$ for some integers a, b, and j such that $a, b \neq 0$ and neither a nor b is a multiple of p, then

$$|x|_p = p^{-j}. \tag{1.2.5}$$

One can check that this defines an ultrametric absolute value function on the field $\mathbf{Q}$ of rational numbers. This corresponds to (c) on p2 of [33], with $\gamma = 1/p$, as mentioned on p3 of [33]. This also corresponds to Definition 2.1.4 on p24 of [70] and Example (2) on p211 of [95], and it is mentioned on p284 of [111]. This is mentioned on p287 of [124] as well, using slightly different terminology.

1.3 Quasimetric absolute value functions

Let k be a field, and let $|\cdot|$ be a nonnegative real-valued function defined on k again. If $|\cdot|$ satisfies (1.2.1) and (1.2.2), then

$$|1_k| = 1, \tag{1.3.1}$$

where 1_k is the multiplicative identity element in k, and $1 = 1_{\mathbf{R}}$ is the multiplicative identity element in $\mathbf{R}$. This follows from the fact that $1_k^2 = 1_k$, so that $|1_k| = |1_k^2| = |1_k|^2$. If $x \in k$ satisfies $x^n = 1_k$ for some positive integer n, then

$$|x| = 1, \tag{1.3.2}$$

because $|x|^n = |x^n| = 1$. In particular,

$$|-1_k| = 1, \tag{1.3.3}$$

because $(-1_k)^2 = 1_k$.

Let us say that $|\cdot|$ is a *quasimetric absolute value function* on k if it satisfies (1.2.1) and (1.2.2), and if there is a positive real number C_1 such that

$$|x + y| \le C_1 \max(|x|, |y|) \tag{1.3.4}$$

for all $x, y \in k$. One can check that (1.3.4) is equivalent to asking that for every $z \in k$ with

$$|z| \le 1, \tag{1.3.5}$$

we have that

$$|1_k + z| \le C_1 \tag{1.3.6}$$

under these conditions. In fact, (1.3.6) corresponds to the condition (iii) in the definition of a *valuation* on p12 of [33]. Note that this holds with $C_1 = 1$

exactly when $|\cdot|$ is an ultrametric absolute value function on k, as in the first part of Lemma 1.3 bis on p15 of [33]. This corresponds to the non-archimedean property in Definition 1.3 on p15 of [33].

If there is a positive real number C_2 such that

$$|x + y| \le C_2\,(|x| + |y|) \tag{1.3.7}$$

for every $x, y \in k$, then (1.3.4) holds, with

$$C_1 = 2\,C_2. \tag{1.3.8}$$

In particular, if $|\cdot|$ is an absolute value function on k, then $|\cdot|$ is a quasimetric absolute value function on k, with $C_1 = 2$. Conversely, (1.3.4) implies (1.3.7), with $C_2 = C_1$.

If $|\cdot|$ is a quasimetric absolute value function on k, then

$$d(x, y) = |x - y| \tag{1.3.9}$$

defines a quasimetric on k. More precisely, one can take C in (1.1.5) equal to C_2 in (1.3.7), or C_0 in (1.1.6) equal to C_1 in (1.3.4). If $|\cdot|$ is an absolute value function on k, then (1.3.9) is a metric on k. If $|\cdot|$ is an ultrametric absolute value function on k, then (1.3.9) is an ultrametric on k. If $|\cdot|$ is the trivial absolute value function on k, then (1.3.9) is the discrete metric on k.

If $k = \mathbf{R}$ or $\mathbf{C}$ and $|\cdot|$ is the standard absolute value function, then (1.3.9) is the standard Euclidean metric. If $k = \mathbf{Q}$ and p is a prime number, then the ultrametric

$$d_p(x, y) = |x - y|_p \tag{1.3.10}$$

associated to the p-adic absolute value function $|\cdot|$ is known as the *p-adic metric* on $\mathbf{Q}$.

1.4 Some helpful inequalities

Let a be a positive real number with $a \le 1$. If r and t are nonnegative real numbers, then it is well known that

$$(r + t)^a \le r^a + t^a. \tag{1.4.1}$$

To see this, observe first that

$$\max(r, t) \le (r^a + t^a)^{1/a}. \tag{1.4.2}$$

This implies that

$$r + t \le \max(r, t)^{1-a}\,(r^a + t^a) \le (r^a + t^a)^{((1-a)/a)+1} = (r^a + t^a)^{1/a}. \tag{1.4.3}$$

This is equivalent to (1.4.1).

If d_X is a metric on a set X, then it follows that

$$d_X(\cdot,\cdot)^a \text{ is a metric on } X. \tag{1.4.4}$$

Similarly, if $|\cdot|$ is an absolute value function on a field k, then

$$|\cdot|^a \text{ is an absolute value function on } k. \tag{1.4.5}$$

This corresponds to the first part of Exercise 1 on p214 of [95].

If d_X is any quasimetric on a set X, then there is a positive real number δ and a metric ρ_X on X such that $d_X(\cdot,\cdot)^\delta$ and $\rho_X(\cdot,\cdot)$ are each bounded by a constant multiple of the other on X. This corresponds to Proposition 14.5 on p110 of [77], and it is also mentioned in the proof of Theorem 2 on p261 of [120].

If $|\cdot|$ is a quasimetric absolute value function on a field k, then the corollary on p14 of [33] says that $|\cdot|^\alpha$ is an absolute value function on k for some positive real number α. We shall say more about this soon.

If $|\cdot|$ is an ultrametric absolute value function on k, then

$$|\cdot|^a \text{ is an ultrametric absolute value function on } k \tag{1.4.6}$$

for every $a > 0$. This corresponds to the second part of Lemma 1.3 bis on p15 of [33], Problem 69 on p43 of [70], and the second part of Exrcise 1 on p214 of [95]. If $|\cdot|$ is a quasimetric absolute value function on k with a constant $C_1 > 0$ as in (1.3.4), then

$$|x+y|^a \le C_1^a \max(|x|^a, |y|^a) \tag{1.4.7}$$

for all $x, y \in k$. This means that $|\cdot|^a$ is a quasimetric absolute value function on k as well, as in Lemma 1.1 on p13 pf [33]. If $|\cdot|$ is the standard absolute value function on $\mathbf{R}$ or $\mathbf{C}$ and $a > 1$, then it is easy to see that $|\cdot|^a$ is not an absolute value function on $\mathbf{R}$ or $\mathbf{C}$, as appropriate.

Suppose that $|\cdot|$ is a quasimetric absolute value function on k with constant C_1 as in (1.3.4) again. Let l be a nonnegative integer, and let $x_1, \ldots, x_{2^l}$ be elements of k. One can check that

$$\left|\sum_{j=1}^{2^l} x_j\right| \le C_1^l \max_{1 \le j \le 2^l} |x_j|, \tag{1.4.8}$$

using induction on l. This corresponds to an argument used in part (ii) of the proof of Lemma 1.2 on p13 of [33]. Of course, there is an analogous statement for a quasimetric on any set.

Let L be a positive integer, and let $x_1, \ldots, x_L$ be elements of k. There is a unique nonnegative integer l such that

$$2^{l-1} < L \le 2^l, \tag{1.4.9}$$

and we can put $x_j = 0$ when $L < j \le 2^l$. Using (1.4.8), we get that

$$\left|\sum_{j=1}^{L} x_j\right| \le C_1^{\log_2 L+1} \max_{1 \le j \le L} |x_j|. \tag{1.4.10}$$

This corresponds to another part of part (ii) of the proof of Lemma 1.2 on p13 of [33].

1.5 The archimedean property

Let k be a field, and let n be a positive integer. If $x \in k$, then let $n \cdot x$ be the sum of n x's in k. Equivalently,

$$n \cdot x = (n \cdot 1_k)\, x. \tag{1.5.1}$$

It is well known and easy to see that $n \mapsto 1_k$ extends to a ring homomorphism from the ring $\mathbf{Z}$ of integers into k.

Let $|\cdot|$ be a quasimetric absolute value function on k. If

$$|n_0 \cdot 1_k| > 1 \tag{1.5.2}$$

for some positive integer n_0, then

$$|n_0^j \cdot 1_k| = |(n_0 \cdot 1_k)^j| = |n_0 \cdot 1_k|^j \to +\infty \text{ as } j \to \infty. \tag{1.5.3}$$

One may say that $|\cdot|$ is *archimedean* on k under these conditions, as on p29 of [70]. The archimedean property is defined in Definitiohn 3 on p213 of [95] in terms of (1.5.2).

Thus we may say that $|\cdot|$ is *non-archimedean* on k if the set of nonnegative real numbers $|n \cdot 1_k|$, where n is an element of the set $\mathbf{Z}_+$ of positive integers, has an upper bound in $\mathbf{R}$. Equivalently, this means that

$$|n \cdot 1_k| \le 1 \tag{1.5.4}$$

for every $n \in \mathbf{Z}_+$. This is the way that the property of being non-archimedean is defined on p213 of [95]. If $|\cdot|$ is an ultrametric absolute value function on k, then it is easy to see that $|\cdot|$ is non-archimedean on k in this sense.

Conversely, if $|\cdot|$ is non-archimedean on k in this sense, then $|\cdot|$ is an ultrametric absolute value function on k. This corresponds to Lemma 1.5 on p16 of [33]. If $|\cdot|$ is an absolute value function on k, then this corresponds to Theorem 2.2.2 on p28 of [70] and Theorem 2 on p213 of [95], and it is also mentioned on p285 of [111]. In fact, the proof in [33] is given for absolute value functions, and another result is used to reduce to this case. Let us suppose for simplicity that $|\cdot|$ is an absolute value function on k, and consider quasimetric absolute value functions afterwards.

Let $x, y \in k$ and $n \in \mathbf{Z}_+$ be given, and note that

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} \cdot x^j\, y^{n-j}, \tag{1.5.5}$$

by the binomial theorem. Of course,

$$\binom{n}{j} \cdot x^j\, y^{n-j} = \left(\binom{n}{j} \cdot 1_k\right) x^j\, y^{n-j} \tag{1.5.6}$$

for each j, as in (1.5.1). This implies that

$$\left|\binom{n}{j} \cdot x^j\, y^{n-j}\right| \le |x|^j\, |y|^{n-j} \tag{1.5.7}$$

for each j, because of (1.5.4). It follows that

$$|x+y|^n = |(x+y)^n| \le \sum_{j=0}^{n} \left| \binom{n}{j} \cdot x^j\, y^{n-j} \right| \tag{1.5.8}$$
$$\le \sum_{j=0}^{n} |x|^j\, |y|^{n-j} \le (n+1)\, \max(|x|, |y|)^n.$$

Using this, we get that

$$|x+y| \le (n+1)^{1/n}\, \max(|x|.|y|). \tag{1.5.9}$$

One can take the limit as $n \to \infty$ on the right side to obtain the ultrametric version of the triangle inequality.

One could also start here with only the hypothesis that $|n \cdot 1_k|$ be bounded, instead of (1.5.4). This would lead to an additional constant factor on the right sides of (1.5.7) and (1.5.9). One would also get the nth root of this constant factor on the right side of (1.5.9), which would not affect the limit as $n \to \infty$.

Suppose now that $|\cdot|$ is a quasimetric absolute value function on k with constant $C_1 > 0$, as in Section 1.2. In this case, we would get an extra factor of

$$C_1^{\log_2(n+1)+1} \tag{1.5.10}$$

on the right side of (1.5.8), using (1.4.10) with $L = n+1$ in the second step. This means that we would get an extra factor of

$$C_1^{(\log_2(n+1)+1)/n} \tag{1.5.11}$$

on the right side of (1.5.9). This would not affect the limit as $n \to \infty$, as before.

1.6 More on the triangle inequality

Let k be a field. If

$$|\cdot| \text{ is an absolute value function on } k, \tag{1.6.1}$$

then

$$|x+y| \le 2\, \max(|x|, |y|) \tag{1.6.2}$$

for all $x, y \in k$, as in Section 1.2. Conversely, if $|\cdot|$ is a quasimetric absolute value function on k with constant 2, so that (1.6.2) holds, then (1.6.1) holds. This corresponds to Lemma 1.2 on p13 of [33].

As a refinement of this, let $|\cdot|$ be a quasimetric absolute value function on k with constant C_1, and suppose that there is a positive real number C_3 such that

$$|n \cdot 1_k| \le C_3\, n \tag{1.6.3}$$

for every positive integer n. Under these conditions, one can show that (1.6.1) holds, using the same argument as in [33].

Before getting to that, note that (1.6.2) implies (1.6.3), with $C_3 = 2$, as in [33]. This follows from (1.3.1) and (1.4.10), with $L = n$ and $x_j = 1_k$ for each $j = 1, \ldots, n$. Of coourse, (1.6.1) implies (1.6.3), with $C_3 = 1$.

Let $|\cdot|$ be a quasimetric absolute value function on k with constant C_1, and let $x, y \in k$ and $n \in \mathbf{Z}_+$ be given. Observe that

$$\left|\binom{n}{j} \cdot x^j\, y^{n-j}\right| = \left|\binom{n}{j} \cdot 1_k\right| |x|^j\, |y|^{n-j} \tag{1.6.4}$$

for each $j = 0, 1, \ldots, n$, by (1.5.6). If (1.6.3) holds, then we get that

$$\left|\binom{n}{j} \cdot x^j\, y^{n-j}\right| \le C_3 \binom{n}{j} |x|^j\, |y|^{n-j} \tag{1.6.5}$$

for each j.

We also have that

$$|x+y|^n = |(x+y)^n| \le C_1^{\log_2(n+1)+1} \max_{0 \le j \le n} \left|\binom{n}{j} \cdot x^j\, y^{n-j}\right|, \tag{1.6.6}$$

using the binomial theorem and (1.4.10), with $L = n + 1$. It follows that

$$|x+y|^n \le C_1^{\log_2(n+1)+1}\, C_3 \max_{0 \le j \le n} \binom{n}{j} |x|^j\, |y|^{n-j}, \tag{1.6.7}$$

by (1.6.5).

This implies that

$$|x+y|^n \le C_1^{\log_2(n+1)+1}\, C_3 \sum_{j=0}^{n} \binom{n}{j} |x|^j\, |y|^{n-j}. \tag{1.6.8}$$

This means that

$$|x+y|^n \le C_1^{\log_2(n+1)+1}\, C_3\, (|x| + |y|)^n, \tag{1.6.9}$$

by the binomial theorem.

Equivalently,

$$|x+y| \le C_1^{(\log_2(n+1)+1)/n}\, C_3^{1/n}\, |x+y| \tag{1.6.10}$$

for every positive integer n. One can take the limit as $n \to \infty$ on the right side to get the usual version of the triangle inequality for $|\cdot|$ on k.

If $|\cdot|$ is a quasimetric absolute value function on k with constant C_1 and a is a positive real number, then $|\cdot|^a$ is a quasimetric absolute value function on k with constant C_1^a, as in Section 1.4. If

$$C_1^a \le 2, \tag{1.6.11}$$

then $|\cdot|^a$ is an absolute value function on k, as before. This holds when a is sufficiently small, as in the corollary on p14 of [33].

Suppose that k has characteristic 0, so that there is a natural embedding of the field $\mathbf{Q}$ of rational numbers into k. If $|\cdot|$ is a quasimetric absolute value function on k, then we get an induced quasimetric absolute value function on $\mathbf{Q}$. Of course, if (1.6.1) holds, then we get an absolute value function on $\mathbf{Q}$. Conversely, if $|\cdot|$ is a quasimetric absolute value function on k, and the induced quasimetric absolute value function on $\mathbf{Q}$ is an absolute value function on $\mathbf{Q}$, then (1.6.3) holds with $C_3 = 1$. This implies that (1.6.1) holds, as before.

1.7 Open and closed balls

Let X be a set, and let $d(\cdot,\cdot)$ be a quasimetric on X. If $x \in X$ and r is a positive real number, then the *open ball* in X centered at x with radius r with respect to $d(\cdot,\cdot)$ is defined by

$$B(x,r) = B_d(x,r) = \{y \in X : d(x,y) < r\}, \tag{1.7.1}$$

as usual. Similarly, if r is a nonnegative real number, then the *closed ball* in X cenetered at x with radius r with respect to $d(\cdot,\cdot)$ is defined by

$$\overline{B}(x,r) = \overline{B}_d(x,r) = \{y \in X : d(x,y) \le r\}. \tag{1.7.2}$$

A subset U of X is said to be an *open set* with respect to $d(\cdot,\cdot)$ if for every $x \in U$ there is an $r > 0$ such that

$$B(x,r) \subseteq U, \tag{1.7.3}$$

as usual. This defines a topology on X, by standard arguments.

If a is a positive real number, then $d(\cdot,\cdot)^a$ is a quasimetric on X too, as in Section 1.1. Observe that

$$B_{d^a}(x,r^a) = B_d(x,r) \tag{1.7.4}$$

for every $x \in X$ and $r > 0$, and similarly that

$$\overline{B}_{d^a}(x,r^a) = \overline{B}_d(x,r) \tag{1.7.5}$$

for every $r \ge 0$. In particular, the topology determined on X by $d(\cdot,\cdot)^a$ is the same as the topology determined by $d(\cdot,\cdot)$.

If $d(\cdot,\cdot)$ is a metric on X, then open balls in X with respect to d are open sets, and closed balls in X with respect to d are closed sets, by standard arguments. This also works when $d(\cdot,\cdot)^a$ is a metric on X for some $a > 0$, by the remarks in the preceding paragraph. A quasimetric associated to a quasimetric absolute value function on a field is of this type, as in the previous section.

Suppose now that $d(\cdot,\cdot)$ is an ultrametric on X. If r is a positive real number, then it is easy to see that

$$d(x,y) < r \tag{1.7.6}$$

defines an equivalence relation on X. The corresponding equivalence classes in X are the same as the open balls in X of radius r with respect to $d(\cdot, \cdot)$. In particular, the complement of an open ball in X of radius r is a union of open balls in X of radius r. This implies that open balls in X are closed sets in X.

Similarly, if r is a nonnegative real number, then

$$d(x, y) \leq r \tag{1.7.7}$$

defines an equivalence relation on X. The corresponding equivalence class in X are the same as the closed balls in X of radius r with respect to $d(\cdot, \cdot)$.

Of course, if $r > 0$, then (1.7.6) implies (1.7.7). This means that equivalence classes in X with respect to (1.7.7) are partitioned into equivalence classes in X with respect to (1.7.6). Equivalently, closed balls in X of radius r are partitioned into open balls of radius r. It follows that closed balls in X of positive radius are open sets in X.

1.8 Equivalent absolute value functions

Let k be a field, and let $|\cdot|_1$, $|\cdot|_2$ be quasimetric absolute value functions on k. Let us say that $|\cdot|_1$ and $|\cdot|_2$ are *equivalent* on k if there is a positive real number a such that

$$|x|_2 = |x|_1^a \tag{1.8.1}$$

for every $x \in k$. Of course, this is the same as saying that

$$|x|_1 = |x|_2^{1/a} \tag{1.8.2}$$

for every $x \in k$. This corresponds to Definition 1.2 on p13 of [33].

Note that (1.8.1) implies that

$$|x - y|_2 = |x - y|_1^a \tag{1.8.3}$$

for all $x, y \in k$. It follows that the corresponding quasimetrics

$$|x - y|_1 \text{ and } |x - y|_2 \text{ determine the same topology on } k, \tag{1.8.4}$$

as in the previous section.

It is easy to see that $w \in k$ satisfies $|w|_1 < 1$ if and only if

$$w^j \to 0 \text{ as } j \to \infty \tag{1.8.5}$$

with respect to the topology determined by $|x - y|_1$. Of course, the analogous statement for $|\cdot|_2$ holds for the same reason. If (1.8.4) holds, then we get that

$$\{w \in k : |w|_1 < 1\} = \{w \in k : |w|_2 < 1\}. \tag{1.8.6}$$

It is easy to see that (1.8.6) holds if and only if

$$\{z \in k : |z|_1 > 1\} = \{z \in k : |z|_2 > 1\}, \tag{1.8.7}$$

by taking $w = 1/z$. In this case, we also have that

$$\{u \in k : |u|_1 = 1\} = \{u \in k : |u|_2 = 1\}. \tag{1.8.8}$$

It is well known that (1.8.6) implies that $|\cdot|_1$ and $|\cdot|_2$ are equivalent on k, and we shall say more about this in Section 1.10. This means that the equivalence of $|\cdot|_1$ and $|\cdot|_2$ on k is equivalent to each of (1.8.4) and (1.8.6). This corresponds to Lemma 3.2 on p20 of [33], and to Lemma 3.1.2 on p42 of [70]. In fact, equivalence of absolute value functions is defined in Definition 3.1.1 on p42 of [70] in terms of (1.8.4).

If (1.8.6) holds, then it is easy to see that for every $u, v \in k$,

$$|u|_1 > |v|_1 \text{ if and only if } |u|_2 > |v|_2. \tag{1.8.9}$$

This is clear when $v = 0$, and otherwise one can take $w = u/v$. Conversely, (1.8.9) implies (1.8.6), by taking $u = 1_k$. Equivalence of absolute value functions is defined (using different terminology) in Definition 2 on p212 of [95] in terms of (1.8.9). Theorem 1 on p212 of [95] states that this implies the equivalence of $|\cdot|_1$ and $|\cdot|_2$ as defined here.

One may also say that $|\cdot|_1$ and $|\cdot|_2$ are *dependent* on k if (1.8.4) holds, as on p283 of [111]. Otherwise, one may say that $|\cdot|_1$ and $|\cdot|_2$ are *independent*, as in [111].

1.9 Nontrivial absolute value functions

Let k be a field, and let $|\cdot|$ be a quasimetric absolute value function on k. Note that $|\cdot|$ is not the trivial absolute value function on k if and only if there is an $x_0 \in k$ such that $x_0 \neq 0$ and $|x_0| \neq 1$. This implies that there are $y_0, z_0 \in k$ such that

$$0 < |y_0| < 1 \text{ and } |z_0| > 1, \tag{1.9.1}$$

using x_0 and $1/x_0$.

In particular,

$$|y_0^j| = |y_0|^j \to 0 \text{ as } j \to \infty \tag{1.9.2}$$

in this case. It follows that the topology determined on k by the quasimetric $|x - y|$ is not the discrete topology under these conditions.

Conversely, if $|\cdot|$ is the trivial absolute value function on k, then $|x - y|$ is the discrete metric on k, as in Section 1.2, and the corresponding topology on k is the discrete topology.

Suppose that $|\cdot|$ is nontrivial on k again, and let $y_0 \in k$ be as in (1.9.1). If $x \in k$, then the first part of Exercise 4 on p214 of [95] states that $|x| \leq 1$ if and only if

$$|y_0\, x^n| < 1 \tag{1.9.3}$$

for every $n \in \mathbf{Z}_+$. Similarly, $|x| = 1$ if and only if $x \neq 0$ and (1.9.3) holds for every $n \in \mathbf{Z}$.

Let $|\cdot|_1$, $|\cdot|_2$ be quasimetric absolute value functions on k. If $|\cdot|_1$ is the trivial absolute value function on k and (1.8.6) holds, then it is easy to see that $|\cdot|_2$ is the trivial absolute value function on k as well, using the remarks at the beginning of the section. This could also be obtained from (1.8.8). This means that (1.8.1) holds for every $a > 0$.

Suppose now that $|\cdot|_1$ is not the trivial absolute value function on k. Suppose also that for every $w \in k$,

$$|w|_1 < 1 \text{ implies that } |w|_2 < 1. \tag{1.9.4}$$

Note that (1.9.4) holds when

(1.9.5) the topology determined on k by $|x-y|_1$ is at least as strong as the topology determined on k by $|x-y|_2$.

Indeed, if this condition holds, and (1.8.5) holds with respect to the topology determined on k by $|x-y|_1$, then (1.8.5) holds with respect to the topology determined on k by $|x-y|_2$.

Using (1.9.4), we get that for every $z \in k$,

$$|z|_1 > 1 \text{ implies that } |z|_2 > 1, \tag{1.9.6}$$

by taking $w = 1/z$.

We would like to show that for every $u \in k$,

$$|u|_1 = 1 \text{ implies that } |u|_2 = 1, \tag{1.9.7}$$

which is another part of Exercise 4 on p214 of [95]. This can be obtained from the characterization of this condition mentioned earlier. An analogous argument is used in the first part of the proof of Lemma 3.1 on p18 of [33].

This means that (1.8.6), (1.8.7) and (1.8.8) hold, as in [33]. If one can show that $|\cdot|_1$ and $|\cdot|_2$ are equivalent on k when (1.8.6) holds, then it follows that $|\cdot|_1$ and $|\cdot|_2$ are equivalent on k when $|\cdot|_1$ is nontrivial on k and (1.9.4) holds, as in Exercise 4 on p214 of [95]. This is the same as Lemma 3.1 on p18 of [33].

Alternatively, (1.9.6) is the same as saying that for each $z \in k$,

$$|z|_2 \le 1 \text{ implies that } |z|_1 \le 1. \tag{1.9.8}$$

Although this seems to be less precise at first, one can use (1.9.4), (1.9.6), and (1.9.8) to get that $|\cdot|_1$ and $|\cdot|_2$ are equivalent on k when $|\cdot|_1$ is nontrivial on k, as in the proof of Proposition 1 on p283 of [111]. This is discussed in the next section.

1.10 Getting equivalence of absolute values

Let k be a field, and let $|\cdot|_1$ and $|\cdot|_2$ be quasimetric absolute value functions on k. We would like to show that $|\cdot|_1$ and $|\cdot|_2$ are equivalent on k when (1.8.6) holds, as mentioned in Section 1.8.

If $|\cdot|_1$ is the trivial absolute value function on k, then $|\cdot|_2$ is the trivial absolute value function on k as well, as mentioned in the previous section. Thus we suppose from now on in this section that $|\cdot|_1$ is not the trivial absolute value function on k.

In this case, it suffices to ask that (1.9.4) hold, instead of (1.8.6). This corresponds to Lemma 3.1 on p18 of [33], part of Exercise 4 on p214 of [95], and Proposition 1 on p283 of [111]. Note that (1.9.4) implies that $|\cdot|_2$ is also nontrivial on k. Although (1.9.4) implies (1.8.6) under these conditions, as in the previous section, it will be enough to use (1.9.4), (1.9.6), and (1.9.8) here. This corresponds to the argument in [111], as before.

Because $|\cdot|_1$ is nontrivial on k, there is a $z_1 \in k$ such that

$$|z_1|_1 > 1, \tag{1.10.1}$$

as in (1.9.1). This implies that

$$|z_1|_2 > 1, \tag{1.10.2}$$

as in (1.9.6). It follows that there is a unique positive real number a such that

$$|z_1|_2 = |z_1|_1^a. \tag{1.10.3}$$

We would like to show that (1.8.1) holds for all $x \in k$. This is clear when $x = 0$, and so we may suppose that $x \neq 0$.

There is a unique real number α such that

$$|x|_1 = |z_1|_1^\alpha, \tag{1.10.4}$$

because of (1.10.1). Let m, n be integers with

$$\alpha < m/n \tag{1.10.5}$$

and $n > 0$. This implies that

$$|x|_1 < |z_1|_1^{m/n}, \tag{1.10.6}$$

so that $|x^n|_1 = |x|_1^n < |z_1|_1^m = |z_1^m|_1$, and thus

$$|x^n/z_1^m|_1 < 1. \tag{1.10.7}$$

It follows that

$$|x^n/z_1^m|_2 < 1, \tag{1.10.8}$$

by (1.9.4), so that $|x|_2^n < |z_1|_2^m$, and thus

$$|x|_2 < |z_1|_2^{m/n}. \tag{1.10.9}$$

This means that

$$|x|_2 \leq |z_1|_2^\alpha = |z_1|_1^{\alpha\, a} = |x|_1^a, \tag{1.10.10}$$

using (1.10.3) in the second step, and (1.10.4) in the third step.

Similarly, there is a unique real number β such that

$$|x|_2 = |z_1|_2^{\beta}, \tag{1.10.11}$$

because of (1.10.2). Let m, n be integers with

$$\beta \le m/n \tag{1.10.12}$$

and $n > 0$, although one could ask that the inequality be strict here as well. This implies that

$$|x|_2 \le |z_1|_2^{m/n}, \tag{1.10.13}$$

so that $|x^n|_2 = |x|_2^n \le |z_1|_2^m = |z_1^m|_2$, and thus

$$|x^n/z_1^m|_2 \le 1. \tag{1.10.14}$$

Using (1.9.8), we get that

$$|x^n/z_1^m|_1 \le 1, \tag{1.10.15}$$

so that $|x|_1^n \le |z_1|_2^m$, and thus

$$|x_1|_1 \le |z_1|_1^{m/n}. \tag{1.10.16}$$

It follows that

$$|x|_1 \le |z_1|_1^{\beta} = |z_1|_2^{\beta/a} = |x|_2^{1/a}, \tag{1.10.17}$$

using (1.10.3) in the second step, and (1.10.11) in the third step.

Of course, (1.8.1) follows from (1.10.10) and (1.10.17).

1.11 Absolute value functions on Q

Let $|\cdot|$ be an absolute value function on $\mathbf{Q}$. A famous theorem of Ostrowski states that $|\cdot|$ is either the trivial absolute value function on $\mathbf{Q}$, or $|\cdot|$ is equivalent to the standard absolute value function on $\mathbf{Q}$, or $|\cdot|$ is equivalent to the p-adic absolute value on $\mathbf{Q}$ for some prime number p. This corresponds to Theorem 2.1 on p16 of [33], and to Theorem 3.1.3 on p44 of [70]. Note that the standard absolute value function on $\mathbf{Q}$ is sometimes denoted $|\cdot|_\infty$, as in Example 1 on p22 of [70].

More precisely, if $|\cdot|$ is an archimedean absolute value function on $\mathbf{Q}$, then $|\cdot|$ is equivalent to the standard absolute value function on $\mathbf{Q}$, as in Theorem 3 on p214 of [95]. If $|\cdot|$ is a nontrivial ultrametric absolute value function on $\mathbf{Q}$, then $|\cdot|$ is equivaent to $|\cdot|_p$ for some prime number p.

Note that one can just as well state Ostrowski's theorem for quasimetric absolute value functions on $\mathbf{Q}$, as in [33]. One can reduce to the case of ordinary absolute value functions, as in Section 1.6.

If $|\cdot|$ is archimedean on $\mathbf{Q}$, then $|n| > 1$ for some positive integer n, and one can take n_0 to be the smallest positive integer such that

$$|n_0| > 1, \tag{1.11.1}$$

as on p44 of [70]. Let a be the unique positive real number such that

$$|n_0| = n_0^a. \tag{1.11.2}$$

One would like to show that

$$|x| = |x|_\infty^a \tag{1.11.3}$$

for every $x \in \mathbf{Q}$, where $|x|_\infty$ is the standard absolute value of x, as before. It suffices to do this when x is a positive integer.

One can first show that there is a positive real number C such that

$$|n| \le C\, n^a \tag{1.11.4}$$

for every positive integer n. Note that $|n| \le 1$ when $n < n_0$, by construction. In order to get (1.11.4), one can express any positive integer n as

$$n = \sum_{j=0}^{l} a_j\, n_0^j, \tag{1.11.5}$$

where l is a nonnegative integer, a_j is a nonnegative integer less than n_0 for each j, and $a_l \ne 0$, as on p44 of [70]. More precisely, l is the unique nonnegative integer such that

$$n_0^l \le n < n_0^{l+1}, \tag{1.11.6}$$

as in [70].

Using (1.11.4), we get that

$$|n|^r = |n^r| \le C\, n^{a\,r} \tag{1.11.7}$$

for all positive integers n and r. This means that

$$|n| \le C^{1/r}\, n^a \tag{1.11.8}$$

for all n, r. It follows that

$$|n| \le n^a \tag{1.11.9}$$

for all $n \ge 1$, by taking the limit as $r \to \infty$ on the right side of (1.11.8), as in [70].

To get the oppositve inequality, we observe that

$$n_0^{a\,(l+1)} = |n_0^{l+1}| \le |n_0^{l+1} - n| + |n| \le (n_0^{l+1} - n)^a + |n|, \tag{1.11.10}$$

using (1.11.9) in the third step. One can use this to get that there is a positive real number c such that

$$c\, n^a \le |n| \tag{1.11.11}$$

for every positive integer n, as in [70]. This implies that

$$c\, n^{a\,r} \le |n^r| = |n|^r \tag{1.11.12}$$

for all positive integers n and r, and one can use this to obtain that

$$n^a \le |n| \tag{1.11.13}$$

for all n, as before. This shows that (1.11.3) holds for all $x \in \mathbf{Z}_+$, and thus for all $x \in \mathbf{Q}$.

The non-archimedean case will be discussed in the next section, after some preliminary remarks about ultrametrics.

1.12 The non-archimedean case

Let X be a set, and let $d(\cdot, \cdot)$ be an ultrametric on X. If $x, y, z \in X$ and

$$d(x, y) < d(y, z), \tag{1.12.1}$$

then

$$d(x, z) = d(y, z). \tag{1.12.2}$$

Indeed,

$$d(x, z) \le d(y, z) \tag{1.12.3}$$

when $d(x, y) \le d(y, z)$. If (1.12.1) holds, then one can use the ultrametric version of the triangle inequality

$$d(y, z) \le \max(d(y, x), d(x, z)) \tag{1.12.4}$$

to get that

$$d(y, z) \le d(x, z). \tag{1.12.5}$$

The fact that (1.12.1) implies (1.12.2) may be described by saying that "all triangles are isosceles", as in Corollary 2.3.4 on p32 of [70].

Let k be a field, and let $|\cdot|$ be an ultrametric absolute value function on k. If $u, v \in k$ and

$$|u| < |v|, \tag{1.12.6}$$

then

$$|u - v| = |v|, \tag{1.12.7}$$

as before. This also corresponds to Lemma 1.4 on p15 of [33], and to the first part of Exercise 2 on p214 of [95].

Suppose now that $|\cdot|$ is an ultrametric absolute value function on $\mathbf{Q}$, so that $|n| \le 1$ for every $n \in \mathbf{Z}_+$. If $|n| = 1$ for every $n \in \mathbf{Z}_+$, then it is easy to see that $|\cdot|$ is the trivial absolute value function on $\mathbf{Q}$. Suppose that $|\cdot|$ is not the trivial absolute value function on $\mathbf{Q}$, so that $|n| < 1$ for some $n \in \mathbf{Z}_+$. Let p be the smallest positive integer such that

$$|p| < 1. \tag{1.12.8}$$

It is easy to see that p is a prime number under these conditions.

There is a unique positive real number a such that

(1.12.9) $$|p| = p^{-a}.$$

We would like to show that

(1.12.10) $$|x| = |x|_p^a$$

for every $x \in \mathbf{Q}$. It suffices to show that this holds when x is a positive integer. More precisely, it is enough to check that

(1.12.11) $$|n| = 1$$

when n is a positive integer that is not a multiple of p. Note that this holds when $n < p$, by construction.

If n is not a multiple of p, then

(1.12.12) $$n = p\,l + r$$

for some integers l, r with $l \geq 0$ and $1 \leq r < p$. Observe that

(1.12.13) $$|p\,l| = |p|\,|l| < 1,$$

and $|r| = 1$, as before. Using this, one can obtain (1.12.11) as in (1.12.7). This is the argument on p46 of [70], which corresponds to the proof of (ii) on p18 of [33]. This is also basically the same as the proof of Theorem 4 on p215 of [95], with some differences in the way that it is explained.

1.13 Some remarks about uniform continuity

Let $(X, d_X(\cdot, \cdot))$ be a metric space, and let E be a subset of X. Of course, the restriction of $d_W(x, w)$ to $x, w \in E$ is a metric on E. If $d_X(\cdot, \cdot)$ is an ultrametric on X, then its restriction to E is an ultrametric on E.

Suppose for the moment that E is a dense set in X. If the restriction of $d_X(\cdot, \cdot)$ to E is an ultrametric on E, then one can check that $d_X(\cdot, \cdot)$ is an ultrametric on X.

If X is complete as a metric space with respect to d_X and E is a closed set in X, then one can check that E is complete with respect to the restriction of d_X to E. Conversely, if E is complete with respect to the restriction to d_X to E, then E is a closed set in X. Indeed, if $\{x_j\}_{j=1}^\infty$ is a sequence of elements of E that converges to an element x of X, then $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in X. This implies that $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in E, so that $\{x_j\}_{j=1}^\infty$ converges to an element of E, by hypothesis. It follows that $x \in E$, because the limit of a convergent sequence in a metric space is unique.

Let (Y, d_Y) be another metric space, and suppose for the moment that f is a uniformly continuous mapping from X into Y. If $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in X, then one can check that

(1.13.1) $$\{f(x_j)\}_{j=1}^\infty \text{ is a Cauchy sequence in } Y.$$

In particular, if Y is complete as a metric space with respect to d_Y, then it follows that $\{f(x_j)\}_{j=1}^\infty$ converges to an element of Y.

Suppose for the moment that $\{x_j\}_{j=1}^\infty$ and $\{w_j\}_{j=1}^\infty$ are sequences of elements of X such that

$$\lim_{j\to\infty} d_X(x_j, w_j) = 0. \tag{1.13.2}$$

One can verify that

$$\lim_{j\to\infty} d_Y(f(x_j), f(w_j)) = 0. \tag{1.13.3}$$

In fact, this propery characterizes uniform continuity. If either $\{x_j\}_{j=1}^\infty$ or $\{w_j\}_{j=1}^\infty$ is a Cauchy sequence in X, then it is easy to see that the other is a Cauchy sequence too. In this case, if Y is complete, then (1.13.3) implies that

$$\lim_{j\to\infty} f(x_j) = \lim_{j\to\infty} f(w_j). \tag{1.13.4}$$

Suppose that E is a dense subset of X again, and that f is a uniformly continuous mapping from E into Y, with respect to the restriction of d_X to E. If Y is complete, then it is well known that there is a unique extension of f to a uniformly continuous mapping from X into Y. More precisely, the uniqueness part only uses the continuity of the extension, and does not involve the completeness of Y.

To get the existence part of the extension, let $x \in X$ be given, and let $\{x_j\}_{j=1}^\infty$ be a sequence of elements of E that converges to x. It is well known and easy to see that $\{x_j\}_{j=1}^\infty$ is a Cauchy sequence in X, and we may consider it as a Cauchy sequence in E. It follows that $\{f(x_j)\}_{j=1}^\infty$ converges to an element of Y, as before. If $x \in E$, then $\{f(x_j)\}_{j=1}^\infty$ converges to $f(x)$ in Y, because f is continuous at x. Otherwise, one would like to define $f(x)$ to be the limit of $\{f(x_j)\}_{j=1}^\infty$ in Y.

If $\{w_j\}_{j=1}^\infty$ is another sequence of elements of E that converges to x, then it is easy to see that (1.13.2) holds. This implies that (1.13.4) holds, so that this definition of $f(x)$ does not depend on the particular sequence of elements of E that converges to x.

The uniform continuity of f on E means that for every $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that for every $u, v \in E$ with

$$d_X(u, v) < \delta(\epsilon), \tag{1.13.5}$$

we have that

$$d_Y(f(u), f(v)) < \epsilon. \tag{1.13.6}$$

If $u, v \in X$ satisfy (1.13.5), then one can check that this extension of f to X has the property that

$$d_Y(f(u), f(v)) \le \epsilon. \tag{1.13.7}$$

This implies that this extension of f to X is uniformly continuous as well.

A mapping ϕ from X into Y is said to be an *isometric embedding* if

$$d_Y(\phi(x), \phi(w)) = d_X(x, w) \tag{1.13.8}$$

for every $x, w \in X$. This implies that ϕ is uniformly continuous and one-to-one on X. If $\phi(X) = Y$, then the inverse of ϕ is an isometry from Y onto X. If E is a dense subset of X and Y is complete, then an isometric embedding of E into Y has a unique extension to an isometric embedding of X into Y.

If X is complete and ϕ is an isometric embedding of X into Y, then $\phi(X)$ is a closed set in Y. More precisely, if X is complete, then $\phi(X)$ is complete with respect to the restriction of d_Y to $\phi(X)$. This implies that $\phi(X)$ is a closed set in Y, as before. If $\phi(X)$ is dense in Y, then it follows that $\phi(X) = Y$.

A *completion* of X may be defined as an isometric embedding of X onto a dense set in a complete metric space. It is well known that such a completion always exists. The remarks in the previous paragraphs may be used to show that completions are unique up to a suitable isometric equivalence. Note that a completion of an ultrametric space is an ultrametric space, by a remark near the beginning of the section.

1.14 Continuity of field operations

Let k be a field with an absolute value funtion $|\cdot|$. It is well known that the field operations on k are continuous with respect to the metric associated to $|\cdot|$. This corresponds to Problem 43 on p30 of [70], and to Exercise 1 on p221 of [95].

More precisely, this means that addition and multiplication are continuous as mappings from $k \times k$ into k, with respect to a suitable product metric on $k \times k$, obtained from the metric on k associated to $|\cdot|$. Similarly,

$$x \mapsto x^{-1} \tag{1.14.1}$$

is continuous as a mapping from $k \setminus \{0\}$ onto itself, with respect to the restriction to $k \setminus \{0\}$ of the metric on k associated to $|\cdot|$.

The space $k^2 = k \times k$ of ordered pairs of elements of k may be considered as a two-dimensional vector space over k, with respect to coordinatewise addition and scalar multiplication. One can define suitable norms on this space with respect to $|\cdot|$ on k, and we shall discuss this further in Section 2.2. These norms determine metrics on $k \times k$, and we may use such a metric as in the preceding paragraph.

In fact, it is easy to see that addition on k defines a uniformly continuous mapping from $k \times k$ into k, with respect to such a metric on $k \times k$. One can also check that multiplication on k defines a mapping from $k \times k$ into k whose restriction to any bounded set in $k \times k$ is uniformly continuous with respect to such a metric.

If r is a positive real number, then one can also verify that (1.14.1) is uniformly continuous as a mapping from

$$\{x \in k : |x| \geq r\} \tag{1.14.2}$$

into k, with respect to the metric on k associated to $|\cdot|$ and its restriction to this set.

1.15 Completing fields with absolute values

Let k be a field with an absolute value function $|\cdot|$. If k_0 is a subfield of k, then the restriction of $|\cdot|$ to k_0 is an absolute value function on k_0. If $|\cdot|$ is an ultrametric absolute value function on k, then its restriction to k_0 is an ultrametric absolute value function on k_0.

If k_0 is a dense set in k with respect to the metric associated to $|\cdot|$, and if the restriction of $|\cdot|$ to k_0 is an ultrametric absolute value function on k_0, then $|\cdot|$ is an ultrametric absolute value function on k, as in the analogous remarks for metric spaces near the beginning of Section 1.13. However, the characterization of ultrametric absolute value functions as non-archimedean absolute value functions in Section 1.5 implies that this works without asking that k_0 be dense in k. This corresponds to Corollary 1 to Lemma 1.5 on p16 of [33].

Let k_1 be another field with an absolute value function $|\cdot|_1$, and let ϕ be an embedding of k as a field into k_1. If

$$|\phi(x)|_1 = |x| \tag{1.15.1}$$

for every $x \in k$, then one may say that ϕ is an *isometric embedding* from k into k_1 with respect to these absolute value functions. This means that ϕ is an isometric embedding with respect to the corresponding metrics, as in Section 1.13.

Suppose that k_0 is a subfield of k that is dense in k with respect to the metric associated to $|\cdot|$. Suppose also that ϕ_0 is an embedding of k_0 as a field into k_1 that is an isometry with respect to the restriction of $|\cdot|$ to k_0 and $|\cdot|_1$ on k_1. If k_1 is complete with respect to the metric associated to $|\cdot|_1$, then ϕ_0 has a unique extension to an isometry from k into k_1 as metric spaces, with respect to the metrics associated to their absolute value functions. One can check that this extension is also an embedding of k as a field into k_1, using the continuity of the field operations, as in the previous section.

Suppose that k_1 is complete with respect to the metric associated to $|\cdot|_1$. If ϕ is an embedding of k as a field into k_1 that is an isometry, and if

$$\phi(k) \text{ is dense in } k_1 \tag{1.15.2}$$

with respect to the metric associated to $|\cdot|_1$, then k_1 may be considered as a *completion* of k with respect to ϕ and these absolute value functions. In particular, this is a completion of k as a metric space with respect to the metric associated to $|\cdot|$, and using the metric on k_1 associated to $|\cdot|_1$. Sometimes one may wish to have k be a subfield of k_1, and anyway one may identify k with $\phi(k)$.

One way to get a completion of k in this sense is to start with a completion of k as a metric space, and to show that the field operations may be extended to the completion. The absolute value function on the completion may be defined as the distance to 0, although one should also verify that this satisfies the appropriate properties. This is the approach used in the proof of Theorem 4.1 on p24 of [33]. One can use the uniform continuity properties of the field operations mentioned in the previous sectio here, as well as the results about extensions of uniformly continuous mappings mentioned in Section 1.13.

A standard way to obtain a completion of a metric space is to use a space of equivalence classes of Cauchy sequences in the metric space. Using this construction, the extension of the field operations to the completion can be obtained more directly. This is outlined in Exercise 1 on p31 of [33], and this is the approach used in Section 3.2 of [70], Section 4 of Chapter V of [95], and in the proof of Proposition 2 on p286 of [111]. More precisely, the space of Cauchy sequences in k may be considered a commutative ring in a suitable way, and the space of sequences of elements of k that converge to 0 is an ideal in this ring. The completion of k may be obtained as the quotient of this ring by this ideal.

A completion of k is unique up to a suitable isomorphic isometric equivalence. This can be obtained from some of the previous remarks about extending isometric embeddings into a complete field with an absolute value function.

Of course, $\mathbf{R}$ may be considered as the completion of $\mathbf{Q}$ with respect to the standard absolute value function.

If p is a prime number, then the field $\mathbf{Q}_p$ of *p-adic numbers* may be obtained by completing $\mathbf{Q}$ with respect to the p-adic absolute value $|\cdot|_p$. We consider $\mathbf{Q}$ as a subfield of $\mathbf{Q}_p$, and the corresponding extension of $|\cdot|_p$ to an ultrametric absolute value function on $\mathbf{Q}_p$ is also denoted $|\cdot|_p$, and called the p-adic absolute value on $\mathbf{Q}_p$. The associated metric is called the p-adic metric on $\mathbf{Q}_p$.

Chapter 2

Norms and ultranorms

2.1 Norms on vector spaces

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k. A nonnegative real-valued function N on V is said to be a *norm* on V with respect to $|\cdot|$ on k if it satisfies the following three conditions. First, for each $v \in V$,

$$N(v) = 0 \text{ if and only if } v = 0. \tag{2.1.1}$$

Second, if $v \in V$ and $t \in k$, then

$$N(t\,v) = |t|\,N(v). \tag{2.1.2}$$

Third, if $v, w \in V$, then

$$N(v + w) \le N(v) + N(w). \tag{2.1.3}$$

Of course, this is the usual definition of a norm when $k = \mathbf{R}$ or $\mathbf{C}$, with the standard absolute value function. This definition of a norm corresponds to Definition 2.1 on p115 of [33] when k and $|\cdot|$ are arbitrary. This is also the same as Definition 5.1.1 on p133 of [70], although some additional conditions on k and $|\cdot|$ are considered there.

In the definition of a norm on p287 of [111], (2.1.2) is replaced with

$$N(t\,v) \le |t|\,N(v). \tag{2.1.4}$$

This implies that $N(0) = 0$ when $t = 0$, and if $t \neq 0$, then we have that

$$N(v) = N(t^{-1}\,(t\,v)) \le |t|^{-1}\,N(t\,v). \tag{2.1.5}$$

This means that

$$|t|\,N(v) \le N(t\,v), \tag{2.1.6}$$

so that (2.1.2) holds. Note that $|\cdot|$ is asked to be nontrivial on k in the definition of a norm in [111].

A nonnegative real-valued function N on V is said to be an *ultranorm* on V with respect to $|\cdot|$ on k if it satisfies (2.1.1) and (2.1.2), and if

$$N(v+w) \le \max(N(v), N(w)) \tag{2.1.7}$$

for every $v, w \in V$. Clearly (2.1.7) implies (2.1.3), so that an ultranorm is a norm in particular. If N is a norm on V and $V \neq \{0\}$, then one can use (2.1.2) with $v \neq 0$ to get that $|\cdot|$ is an ultrametric absolute value function on k.

If N is a norm on V, then it is easy to see that

$$d_N(v, w) = N(v - w) \tag{2.1.8}$$

is a metric on V. If V is an ultranorm on V, then (2.1.8) is an ultrametric on V.

We may consider k as a one-dimensional vector spaces over itself, and $|\cdot|$ as a norm on k with respect to itself. This is an ultranorm when $|\cdot|$ is an ultrametric absolute value function on k.

If k_1 is a field that contains k as a subfield, then k_1 may be considered as a vector space over k. If $|\cdot|_1$ is an absolute value function on k_1 that is equal to $|\cdot|$ on k, then $|\cdot|_1$ may be considered as a norm on k_1 with respect to $|\cdot|$ on k. If $|\cdot|_1$ is an ultrametric absolute value function on k_1, then it may be considered as an ultranorm on k_1 as a vector space over k. Remember that this happens exactly when $|\cdot|$ is an ultrametric absolute value function on k, as mentioned near the beginning of Section 1.15.

If k is any field, then we can take $|\cdot|$ to be the trivial absolute value function. Similarly, if V is any vector space over k, then the *trivial ultranorm* may be defined by putting $N(v) = 1$ when $v \neq 0$, and $N(0) = 0$. It is easy to see that this is an ultranorm on V with respect to the trivial absolute value function on k. The corresponding ultrametric (2.1.8) is the same as the discrete metric on V.

2.2 Some norms on k^n

Let k be a field with an absolute value function $|\cdot|$, and let n be a positive integer. Also let k^n be the space of n-tuples of elements of k, considered as a vector space over k with respect to coordinatewise addition and scalar multiplication. If $v = (v_1, \ldots, v_n) \in k^n$, then put

$$\|v\|_\infty = \max(|v_1|, \ldots, |v_n|). \tag{2.2.1}$$

One can check that this defines a norm on k^n with respect to $|\cdot|$ on k. If $|\cdot|$ is an ultrametric absolute value function on k, then this defines an ultranorm on k^n with respect to $|\cdot|$ on k.

If r is a positive real number, then put

$$\|v\|_r = \Big(\sum_{j=1}^n |v_j|^r\Big)^{1/r} \tag{2.2.2}$$

for every $v \in k^n$. It is easy to see that this satisfies the first two conditions in the definition of a norm. It is also easy to see that this satisfies the triangle inequality when $r = 1$.

Suppose for the moment that $k = \mathbf{R}$, with the standard absolute value function. If $r = 2$, then (2.2.2) is the standard Euclidean norm on $\mathbf{R}^n$. If $1 < r < \infty$, then it is well known that (2.2.2) satisfies the triangle inequality, and is thus a norm. This is *Minkowski's inequality* for finite sums.

One can use this to get that (2.2.2) satisfies the triangle inequality on k^n when $r \geq 1$, for arbitrary k and $|\cdot|$. This means that (2.2.2) defines a norm on k^n when $r \geq 1$.

Of course, if $n = 1$, then (2.2.1) and (2.2.2) are the same as the absolute value function on k. If $n \geq 2$, then one can check that (2.2.2) does not satisfy the triangle inequality on k^n when $r < 1$.

If $r \leq 1$ and $v, w \in k^n$, then

$$\begin{aligned} \|v + w\|_r^r &= \sum_{j=1}^n |v_j + w_j|^r \leq \sum_{j=1}^n (|v_j| + |w_j|)^r \\ &\leq \sum_{j=1}^n (|v_j|^r + |w_j|^r) = \|v\|_r^r + \|w\|_w^r, \end{aligned} \tag{2.2.3}$$

where the third step is as in Section 1.4. One can use this to get that

$$\|v - w\|_r^r \tag{2.2.4}$$

defines a metric on k^n. Note that $|\cdot|^r$ is an absolute value function on k in this case as well. In fact, $\|\cdot\|_r^r$ is a norm on k^n with respect to $|\cdot|^r$ on k. This is the same as the analogue of (2.2.2) with $r = 1$ using $|\cdot|^r$ in place of $|\cdot|$.

Observe that

$$\|v\|_\infty \leq \|v\|_r \leq n^{1/r}\, \|v\|_\infty \tag{2.2.5}$$

for every $v \in k^n$ and $r > 0$.

Open and closed balls in k^n with respect to the metric associated to $\|\cdot\|_\infty$ correspond exactly to products of open and closed balls in k with respect to the metric associated to $|\cdot|$, respectively. One can use this to check that the topology determined on k^n by the metric associated to $\|\cdot\|_\infty$ is the same as the product topology, using the topology determined on k by the metric associated to $|\cdot|$. If $r \geq 1$, then the topology determined on k^n by the metric associated to $\|\cdot\|_r$ is the same as the topology determined by the metric associated to $\|\cdot\|_\infty$, because of (2.2.5). Similarly, if $r < 1$, then this is the same as the topology determined on k^n by (2.2.4).

2.3 Products of two vector spaces

Let k be a field with an absolute value function $|\cdot|$, and let V, W be vector spaces over k with norms N_V, N_W with respect to $|\cdot|$ on k, respectively. The

Cartesian product $V \times W$ of V and W may be considered as a vector space over k as well, with respect to coordinatewise addition and scalar multiplication. This corresponds to the direct sum of V and W, as a vector space over k.

If $v \in V$ and $w \in W$, then put

$$\|(v, w)\|_{V \times W, \infty} = \max(N_V(v), N_W(w)). \tag{2.3.1}$$

One can check that this defines a norm on $V \times W$ with respect to $|\cdot|$ on V. If N_V and N_W are ultranorms on V and W, respectively, then this is an ultranorm on $V \times W$.

If r is a positive real number, then

$$\|(v, w)\|_{V \times W, r} = (N_V(v)^r + N_W(w)^r)^{1/r} \tag{2.3.2}$$

satisfies the first two conditions in the definition of a norm. It is easy to see that this satisfies the triangle inequality when $r = 1$, and for $r > 1$ this can be obtained from the triangle inequality for (2.2.2) on $\mathbf{R}^2$. We also have that

$$\|(v, w)\|_{V \times W, \infty} \le \|(v, w)\|_{V \times W, r} \le 2^{1/r} \, \|(v, w)\|_{V \times W, \infty} \tag{2.3.3}$$

for every $r > 0$.

Open and closed balls in $V \times W$ with respect to the metric associated to (2.3.1) corresponds exactly to products of open and closed balls in V and W with respect to the metrics associated to N_V and N_W, respectively. This implies that the topology determined on $V \times W$ by the metric associated to (2.3.1) is the same as the product topology, using the topologies on V and W determined by the metrics associated to N_V and N_W, respectively. This is the same as the topology determined on $V \times W$ by the metric associated to (2.3.2) when $r \ge 1$.

In particular, we can take $V = W$ and $N_V = N_W$, to get some norms and metrics on $V \times V$. It is easy to see that addition on V is uniformly continuous as a mapping from $V \times V$ into V, with respect to such a metric, and the metric on V associated to N_V.

Similarly, we can get some norms and metrics on $k \times V$, using $|\cdot|$ on k. One can check that scalar multiplication on V defines a mapping from $k \times V$ into V that is uniformly continuous on bounded subsets of $k \times V$ with respect to such a metric.

If $v \in V$, then it is easy to see that

$$t \mapsto t\, v \tag{2.3.4}$$

is uniformly continuous as a mapping from k into V, with respect to the metrics associated to $|\cdot|$ and N_V. Similarly, if $t \in k$, then

$$v \mapsto t\, v \tag{2.3.5}$$

is uniformly continuous as a mapping from V into itself, with respect to the metric associated to N_V.

Let a be a positive real number with $a \le 1$, and remember that $|\cdot|^a$ is an absolute value function on k, as in Section 1.4. One can check that $N_V(\cdot)^a$ is a norm on V with respect to $|\cdot|^a$ on k. If $|\cdot|$ is an ultrametric absolute value function on k, then $|\cdot|^a$ is an ultrametric absolute value function on k for every $a > 0$, as before. If N_V is an ultranorm on V with respect to $|\cdot|$ on k, then $N_V(\cdot)^a$ is an ultranorm on V with respect to $|\cdot|^a$ on k for every $a > 0$.

2.4 Completions and isometric linear mappings

Let k be a field with an absolute value function $|\cdot|$, let V be a vector space over k, and let N_V be a norm on V with respect to $|\cdot|$ on k. If V_0 is a linear subspace of V, then the restriction of N_V to V_0 is a norm on V_0 with respect to $|\cdot|$ on k. If N_V is an ultranorm on V, then the restriction of N_V to V_0 is an ultranorm on V_0. If V_0 is dense in V with respect to the metric associated to N_V, and if the restriction of N_V to V_0 is an ultranorm with respect to $|\cdot|$ on k, then one can check that N_V is an ultranorm on V.

Suppose for the moment that V is complete with respect to the metric associated to N_V. If k is not already complete with respect to the metric associated to $|\cdot|$, then one can get a completion of k as in Section 1.15. One can show that scalar multiplication on V can be extended to the completion of k, so that V may be considered as a vector space over the completion of k, and N_V may be considered as a norm on V with respect to the extension of $|\cdot|$ to the completion of k. To do this, one can first observe that for each $v \in V$, (2.3.4) can be extended to a uniformly continuous mapping from the completion of k into V. This extends scalar multiplication on V to the completion of k, and one can check that this extension has the appropriate properties.

Let W be another vector space over k with a norm N_W. A linear mapping T from V into W is said to be *isometry* if

$$N_W(T(v)) = N_V(v)) \tag{2.4.1}$$

for every $v \in V$. This implies that T is an isometric embedding with respect to the corresponding metrics, as in Section 1.13.

Let V_0 be a linear subspace of V that is dense in V with respect to the metric associated to N_V. Also let T_0 be an isometric linear mapping from V_0 into W, with respect to the restriction of N_V to V_0. If W is complete with respect to the metric associated to N_W, then T_0 has a unique extension to an isometry from V into W, with respect to the metrics associated to their norms, as in Section 1.13. One can check that this extension is a linear mapping too, using the continuity of the vector space operations on V and W, as in the previous section.

Suppose that W is complete with respect to the metric associated to N_W. If T is an isometric linear mapping from V into W, and if

$$T(V) \text{ is dense in } W \tag{2.4.2}$$

with respect to the metric associated to N_W, then W may be considered as a *completion* with respect to T and these norms. Of course, this is a completion of V as a metric space in particular, with respect to the metric assocuated to N_V, and using the metric on W associated to N_W. Sometimes one might like to have V be a linear subspace of W, and one can identify V with $T(V)$ anyway.

If V is complete with respect to the metric associated to N_V, then V is said to be a *Banach space* with respect to N_V. Otherwise, one can get a completion that is also a vector space over k with a norm with respect to $|\cdot|$ on k. As before, one can start with a completion of V as a metric space, and show that the vector space operations on V can be extended to the completion. The norm on the completion may be defined as the distance to 0, and one can verify that this has the appropriate properties.

Alternatively, one can show that the space of Cauchy sequences in V is a vector space over k in a suitable way, and that the space of sequences of elements of V converging to 0 is a linear subspace of this vector space. The completion of V may be defined initially as a vector space over k as the corresponding quotient space, and it is not too difficult to define a norm on this quotient in a suitable way.

One can show that s completion of V is unique up to a suitable isometric linear equivalence, using some of the previous remarks about extending isometric linear mappings on a dense linear subspace of a vector space with a norm into a Banach space.

2.5 Equivalent norms

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k. Also let N_1 and N_2 be norms on V with respect to $|\cdot|$ on k. We say that N_1 and N_2 are *equivalent* as norms on V if there are positive real numbers $C_{1,2}$, $C_{2,1}$ such that

$$N_1(v) \le C_{1,2}\, N_2(v) \tag{2.5.1}$$

and

$$N_2(v) \le C_{2,1}\, N_1(v) \tag{2.5.2}$$

for every $v \in V$. This corresponds to Definition 2.2 on p116 of [33]. This also corresponds to Definition 5.1.3 on p135 of [70], and it is mentioned on p287 of [111] as well, although the discussions in [70, 111] include some additional conditions on $|\cdot|$ on k.

Let d_{N_1}, d_{N_2} be the metrics on V corresponding to N_1, N_2, respectively, as in Section 2.1. Note that (2.5.1) is the same as saying that

$$d_{N_1}(v,w) \le C_{1,2}\, d_{N_2}(v,w) \tag{2.5.3}$$

for every $v, w \in V$. This implies that the topology determined on V by d_{N_2} is at least as strong as the topology determined by d_{N_1}. Similarly, (2.5.2) is the same as saying that

$$d_{N_2}(v,w) \le C_{2,1}\, d_{N_1}(v,w) \tag{2.5.4}$$

for every $v, w \in V$. This implies that the topology determined on V by d_{N_1} is at least as strong as the topology determined by d_{N_2}.

Let us use $B_{N_1}(v,r)$ and $B_{N_2}(v,r)$ for the open balls in V centered at $v \in V$ with radius $r > 0$ with respect to d_{N_1} and d_{N_2}, respectively. If the topology determined on V by d_{N_2} is at least as strong as the topology determined by d_{N_1}, then there is a positive real number r_2 such that

$$B_{N_2}(0,r_2) \subseteq B_{N_1}(0,1). \tag{2.5.5}$$

Similarly, if the topology determined on V by d_{N_1} is at least as strong as the topology determined by d_{N_2}, then there is an $r_1 > 0$ such that

$$B_{N_1}(0,r_1) \subseteq B_{N_2}(0,1). \tag{2.5.6}$$

If $t \in k$ and $E \subseteq V$, then put

$$t\,E = \{t\,v : v \in E\}. \tag{2.5.7}$$

If $t \neq 0$ and r is a positive real number, then

$$t\,B_{N_1}(0,r) = B_{N_1}(0,|t|\,r), \tag{2.5.8}$$

and similarly for N_2. Thus (2.5.5) implies that

$$B_{N_2}(0,|t|\,r_2) \subseteq B_{N_1}(0,|t|), \tag{2.5.9}$$

and (2.5.6) implies that

$$B_{N_1}(0,|t|\,r_1) \subseteq B_{N_2}(0,|t|). \tag{2.5.10}$$

Suppose for the moment that $|\cdot|$ is not the trivial absolute value function on k. If (2.5.5) holds, then one can use (2.5.9) to get that (2.5.1) holds for some $C_{1,2} > 0$. Similarly, if (2.5.6) holds, then one can use (2.5.10) to get that (2.5.2) holds for some $C_{2,1} > 0$. In particular, if d_{N_1} and d_{N_2} determine the same topology on V, then it follows that N_1 and N_2 are equivalent on V. This corresponds to Problem 194 on p135 of [70], and to part of the proof of Proposition 3 on p288 of [111].

Suppose that k is complete with respect to the metric associated to $|\cdot|$. If V has finite dimension over k, then it is well known that any two norms on V with respect to $|\cdot|$ on k are equivalent. We also get that V is complete with respect to the metric associated to any norm. This corresponds to Lemma 2.1 on p116 of [33], Theorem 5.2.1 on p137 of [70], and Proposition 3 on p288 of [111]. More precisely, the completeness part corresponds to part of Proposition 4 on p291 of [111], and its proof.

If the dimension of V is equal to one, then any norm on V with respect to $|\cdot|$ on k is a positive multiple of any other norm. It is easy to see that V is complete with respect to the metric associated to any norm on this case, because k is complete, by hypothesis.

We can reduce to the case where $V = k^n$ for some positive integer n, using a basis for V. The proof uses induction on n, and is discussed in the next section.

2.6 More on norms on k^n

Let k be a field with an absolute value function $|\cdot|$, let n be a positive integer, and let N be a norm on k^n with respect to $|\cdot|$ on k. Also let $e_1, \ldots, e_n$ be the standard basis vectors in k^n, so that the lth component of e_j is equal to 1 when $j = l$, and to 0 when $j \neq l$. If $v \in k^n$, then

$$v = \sum_{j=1}^{n} v_j \, e_j, \tag{2.6.1}$$

so that

$$N(v) \le \sum_{j=1}^{n} |v_j| \, N(e_j). \tag{2.6.2}$$

If N is an ultranorm on k^n, then

$$N(v) \le \max_{1 \le j \le n} (|v_j| \, N(e_j)). \tag{2.6.3}$$

Using (2.6.2), we get that

$$N(v) \le \Big(\sum_{j=1}^{n} N(e_j)\Big) \, \|v\|_\infty, \tag{2.6.4}$$

where $\|v\|_\infty$ is as in Section 2.2. If N is an ultranorm on k^n, then

$$N(v) \le \Big(\max_{1 \le j \le n} N(e_j)\Big) \, \|v\|_\infty, \tag{2.6.5}$$

because of (2.6.3).

Suppose from now on in this section that k is complete with respect to the metric associated to $|\cdot|$. We would like to show that there is a positive real number C such that

$$\|v\|_\infty \le C \, N(v) \tag{2.6.6}$$

for every $v \in V$.

It is easy to see that k^n is complete with respect to the metric associated to $\|\cdot\|_\infty$. One can use this to get that k^n is complete with respect to the metric associated to N, once we have (2.6.6).

We may as well suppose that $n \ge 2$, as in the previous section. We can use the induction hypothesis to get that there is a positive real number C_0 such that

$$\|v\|_\infty \le C_0 \, N(v) \tag{2.6.7}$$

for every $v \in k^n$ with $v_n = 0$. We also get that

$$\{v \in k^n : v_n = 0\} \tag{2.6.8}$$

is a complete as a metric space, with respect to the metric associated to the restriction of N to this subspace. This is the same as the restriction to this subspace of the metric on k^n associated to N.

One can use the completeness of (2.6.8) with respect to the restriction of the metric on k^n associated to N to get that (2.6.8) is a closed set in k^n with respect to this metric. One can use this to get that there is a positive real number c such that

$$N(v) \geq c \tag{2.6.9}$$

when $v_n = 1$. This implies that

$$N(v) \geq c\,|v_n| \tag{2.6.10}$$

for every $v \in k^n$. One can use this and (2.6.7) to get that (2.6.6) holds for some $C > 0$.

This means that N is equivalent to $\|\cdot\|_\infty$ on k^n, because of (2.6.4). It follows that any two norms on k^n are equivalent.

2.7 Corollaries about uniqueness and completeness

Let k_0 be a field, and let $|\cdot|_1$, $|\cdot|_2$ be absolute value functions on k_0. Suppose that there is a positive real number $C_{1,2}$ such that

$$|x|_1 \leq C_{1,2}\,|x|_2 \tag{2.7.1}$$

for every $x \in k_0$. This implies that

$$|x|_1^n = |x^n|_1 \leq C_{1,2}\,|x^n|_2 = C_{1,2}\,|x|_2^n \tag{2.7.2}$$

for every $x \in k_0$ and positive integer n, so that

$$|x|_1 \leq C_{1,2}^{1/n}\,|x|_2. \tag{2.7.3}$$

It follows that

$$|x|_1 \leq |x|_2 \tag{2.7.4}$$

for every $x \in k_0$, by taking the limit as $n \to \infty$ on the right side of (2.7.3). Of course, the same type of argument was used in Section 1.11.

Similarly, if there is a posiitve real number $C_{2,1}$ such that

$$|x|_2 \leq C_{2,1}\,|x|_1 \tag{2.7.5}$$

for every $x \in k_0$, then

$$|x|_2 \leq |x|_1 \tag{2.7.6}$$

for every $x \in k_0$. If there are positive real numbers $C_{1,2}$ and $C_{2,1}$ such that (2.7.1) and (2.7.5) hold for every $x \in k_0$, then we get that

$$|x|_1 = |x|_2 \tag{2.7.7}$$

for every $x \in k_0$. This is analogous to the discussion of equivalent absolute value functions starting in Section 1.8, with stronger hypotheses and some simplifications.

Let k be a field with an absolute value function $|\cdot|$, and suppose that k is complete with respect to the metric associated to $|\cdot|$. Also let k_1 be a field that contains k as a subfield, and suppose that k_1 is a *finite extension* of k, so that k_1 has finite dimension as a vector space over k. Under these conditions, there is at most one absolute value function on k_1 that agrees with $|\cdot|$ on k. This is Corollary 1 on p118 of [33], and part of the uniqueness part of Proposition 4 on p291 of [111]. This corresponds to Corollary 5.3.2 on p143 of [70] when $k = \mathbf{Q}_p$ for some prime number p, with the p-adic absolute value function.

Indeed, an absolute value function on k_1 that agrees with $|\cdot|$ on k may be considered as a norm on k_1, as a vector space over k, with respect to $|\cdot|$ on k. Any two such absolute value functions on k_1 are equivalent as norms, as in the previous two sections. This implies that these two absolute value functions are the same on k_1, by the remarks at the beginning of the section.

Of course, if $|\cdot|$ is the trivial absolute value function on k, then the trivial absolute value function on k_1 is equal to $|\cdot|$ on k. In this case, we get that any absolute value function on k_1 that is equal to the trivial absolute value function on k is the trivial absolute value function on k_1. Another approach to this will be mentioned in the next section.

If $|\cdot|$ is nontrivial on k, then the uniqueness of an extension of $|\cdot|$ to an absolute value function on k_1 is part of Theorem 15 on p258 of [95] as well. The argument used there is somewhat different in some ways, and this will be discussed further in Section 2.9.

One could also use the fact that the metrics associated to any two absolute value functions on k_1 that are equal to $|\cdot|$ on k determine the same topology on k_1, because they are equivalent as norms on k_1 with respect to $|\cdot|$ on k, as before. This implies that they are equivalent as absolute value functions on k_1, as in Section 1.8. One can use this to get that the two absolute value functions on k_1 are the same when $|\cdot|$ is nontrivial on k, and otherwise one can observe that they are both equivalent to the trivial absolute value function on k_1.

Let $|\cdot|_1$ be an absolute value function on k_1 that is equal to $|\cdot|$ on k. Note that k_1 is complete with respect to the metric associated to $|\cdot|_1$, as in the previous two sections. This is Corollary 2 on p118 of [33], and part of Proposition 4 on p291 of [111]. If $|\cdot|$ is nontrivial on k, then this is the same as the consequence (2) on p258 of [95] of Lemma 1 on p257 of [95], and it is mentioned again in Theorem 17 on p262 of [95]. This also corresponds to part of Proposition 5.3.1 on p142 of [70] when $k = \mathbf{Q}_p$ for some prime number p, with the p-adic absolute value function.

2.8 Another uniqueness argument

Let k be a field with an ultrametric absolute value function $|\cdot|$, and let $x_1, \ldots, x_n$ be finitely many elements of k such that

$$\sum_{j=1}^{n} x_j = 0. \tag{2.8.1}$$

Under these conditions, the second part of Exercise 2 on p214 of [95] states that

$$|x_j| = |x_l| \tag{2.8.2}$$

for some $j \neq l$. Otherwise, one could show that

$$\left|\sum_{j=1}^{n} x_j\right| = \max(|x_1|, \ldots, |x_n|), \tag{2.8.3}$$

using a remark in Section 1.12.

Now let k_1 be a field, let k_0 be a subfield of k, and let $|\cdot|_1$ be an absolute value function on k_1. Suppose that the restriction of $|\cdot|_1$ to k_0 is the trivial absolute value function on k_0. This implies that $|\cdot|_1$ is non-archimedean on k_1, so that $|\cdot|_1$ is an ultrametric absolute value function on k_1, as in Section 1.5.

Suppose that $x \in k_1$ is *algebraic* over k_0, so that x satisfies a nontrivial polynomial equation with coefficients in k_0. This means that there are a positive inetger m and elements $a_0, a_1, \ldots, a_m$ of k_0 such that

$$\sum_{j=0}^{m} a_j\, x^j = 0. \tag{2.8.4}$$

We would like to check that

$$|x|_1 = 1 \tag{2.8.5}$$

when $x \neq 0$.

If we leave out the terms on the left side of (2.8.4) where $a_j = 0$, then we get a sum of n terms of the form $a_j\, x^j$ that is equal to 0 for some $n \leq m$. The remark at the beginning of the section implies that

$$|a_j\, x^j|_1 = |a_l\, x^l|_1 \tag{2.8.6}$$

for some j, l with $j \neq l$ and $a_j, a_l \neq 0$. This implies that

$$|x|_1^j = |x|_1^l, \tag{2.8.7}$$

because $|a_j|_1 = |a_l|_1 = 1$, by hypothesis. It follows that (2.8.5) holds when $x \neq 0$.

Suppose that k_1 is an *algebraic extension* of k_0, so that every element of k_1 is algebraic over k_0. In this case, we get that $|\cdot|_1$ is the trivial abslute value function on k_1 as well. This is Exercise 3 on p214 of [95].

2.9 More on completeness

Let k be a field with an absolute value function $|\cdot|$, and suppose that k is complete with respect to the metric associated to $|\cdot|$. Also let k_1 be a field that contains k as a subfield, and let $|\cdot|_1$ be an absolute value function on k_1 that agrees with $|\cdot|$ on k. Remember that k_1 may be considered as a vector space over k, and let $u_1, \ldots, u_n$ be finitely many elements of k_1 that are linearly independent in k_1, as a vector space over k.

Let $\{v_j\}_{j=1}^\infty$ be a sequence of elements of the linear span of $u_1, \ldots, u_n$ in k_1, as a vector space over k. Thus

$$v_j = \sum_{l=1}^n v_{j,l}\, u_l \tag{2.9.1}$$

for each j, where $v_{j,l} \in k$ for $l = 1, \ldots, n$ are uniquely determined by v_j. If $|\cdot|$ is nontrivial on k, then Lemma 1 on p257 of [95] states that $\{v_j\}_{j=1}^\infty$ is a Cauchy sequence in k_1 with respect to the metric associated to $|\cdot|_1$ if and only if $\{v_{j,l}\}_{j=1}^\infty$ is a Cauchy sequence in k with respect to the metric associated to $|\cdot|$ for each $l = 1, \ldots, n$.

Note that

$$\phi(w) = \sum_{l=1}^n w_l\, u_l \tag{2.9.2}$$

defines a one-to-one linear mapping from k^n into k_1, as vector spaces over k. Using this, it is easy to see that

$$N(w) = |\phi(w)|_1 \tag{2.9.3}$$

is a norm on k^n with respect to $|\cdot|$ on k. We also have that N si equivalent to the usual norm $\|\cdot\|_\infty$ on k^n, as in Section 2.6.

Let $\{w_j\}_{j=1}^\infty$ be a sequence of elements of k^n. The equivalence of N and $\|\cdot\|_\infty$ on k^n implies that $\{w_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to the metric on k^n associated to the norm $\|\cdot\|_\infty$ if and only if $\{w_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to the metric on k^n associated to N. This means that $\{w_j\}_{j=1}^\infty$ is a Cauchy sequence with respect to the metric on k^n associated to $\|\cdot\|_\infty$ if and only if $\{\phi(w_j)\}_{j=1}^\infty$ is a Cauchy sequence in k_1 with respect to the metric associated to $|\cdot|_1$. This is equivalent to the conclusion of Lemma 1 on p257 of [95].

Similarly, $\{w_j\}_{j=0}^\infty$ converges to 0 with respect to the metric on k^n associated to $\|\cdot\|_\infty$ if and only if $\{w_j\}_{j=1}^\infty$ converges to 0 with respect to the metric on k^n associated to N. This means that $\{w_j\}_{j=1}^\infty$ converges to 0 with respect to the metric on k^n associated to $\|\cdot\|_\infty$ if and only if $\{\phi(w_j)\}_{j=1}^\infty$ converges to 0 in k_1 with respect to the metric associated to $|\cdot|_1$. The "if" part of this statement corresponds to the consequence (1) on p258 of [95] of Lemma 1 on p257 of [95].

Suppose now that k_1 is a finite extension of k, so that we can take $u_1, \ldots, u_n$ to be a basis for k_1, as a vector space over k. In this case, consequence (2) on p258 of [95] of Lemma 1 on p257 of [95] says that k_1 is complete with respect to the metric associated to $|\cdot|_1$, as mentioned in Section 2.7. Alternatively, k^n is complete with respect to the metric associated to $\|\cdot\|_\infty$, as mentioned in Section 2.6, so that k^n is complete with respect to the metric associated to N. This implies that k_1 is complete with respect to the metric associated to $|\cdot|_1$, because $\phi(k^n) = k_1$ under these conditions, as before.

The uniqueness of $|\cdot|_1$ on k_1 is part of Theorem 15 on p258 of [95], as mentioned in Section 2.7. The proof uses consequence (1) of Lemma 1 on p257 of [95] mentioned earlier. The proof uses some other properties of field extensions too, and we shall return to those later.

2.10 Infinite series

Let k be a field with an absolute value function $|\cdot|$, and let V be a vector space over k with a norm N with respect to $|\cdot|$ on k. An infinite series $\sum_{j=1}^\infty v_j$ of elements of V is said to *converge* with respect to N if the corresponding sequence of partial sums

$$\sum_{j=1}^n v_j \tag{2.10.1}$$

converges in V with respect to the metric associated to N. In this case, the sum of the series is defined as an element of V by

$$\sum_{j=1}^\infty v_j = \lim_{n\to\infty} \sum_{j=1}^n v_j, \tag{2.10.2}$$

as usual. If $t \in k$, then it is easy to see that $\sum_{j=1}^\infty t\, v_j$ converges as well, with

$$\sum_{j=1}^\infty t\, v_j = t \sum_{j=1}^\infty v_j. \tag{2.10.3}$$

If $\sum_{j=1}^\infty w_j$ is another convergent series of elements of V, then $\sum_{j=1}^\infty (v_j + w_j)$ converges too, with

$$\sum_{j=1}^\infty (v_j + w_j) = \sum_{j=1}^\infty v_j + \sum_{j=1}^\infty w_j. \tag{2.10.4}$$

It is easy to see that the sequence of partial sums (2.10.1) is a Cauchy sequence in V with respect to the metric associated to N if and only if for every $\epsilon > 0$ there is a positive integer L such that

$$N\Big(\sum_{j=l}^n v_j\Big) < \epsilon \tag{2.10.5}$$

when $n \ge l \ge L$. A necessary condition for this to hold is that

$$\lim_{j\to\infty} N(v_j) = 0, \tag{2.10.6}$$

by taking $l = n$ in (2.10.5). This is the same as saying that $\{v_j\}_{j=1}^\infty$ converges to 0 in V, with respect to the metric associated to N. Note that

$$N\Big(\sum_{j=l}^{n} v_j\Big) \le \sum_{j=l}^{n} N(v_j) \tag{2.10.7}$$

for all $n \ge l \ge 1$.

If

$$\sum_{j=1}^{\infty} N(v_j) \tag{2.10.8}$$

converges as an infinite series of nonnegative real numbers, then $\sum_{j=1}^\infty v_j$ is said to converge *absolutely* with respect to N. This implies that the sequence of partial sums (2.10.1) is a Cauchy sequence in V with respect to the metric associated to N, because of the remarks in the previous paragraph. If V is a Banach space with respect to N, then it follows that $\sum_{j=1}^\infty v_j$ converges in V. Under these conditions, we also have that

$$N\Big(\sum_{j=1}^{\infty} v_j\Big) \le \sum_{j=1}^{\infty} N(v_j). \tag{2.10.9}$$

Suppose now that N is an ultranorm on V with respect to $|\cdot|$ on k. This means that

$$N\Big(\sum_{j=l}^{n} v_j\Big) \le \max_{l \le j \le n} N(v_j) \tag{2.10.10}$$

for all $n \ge l \ge 1$. If (2.10.6) holds, then it is easy to see that the sequence of partial sums (2.10.1) is a Cauchy sequence in V with respect to the metric associated to N, using (2.10.10) and the earlier characterization in terms of (2.10.5). If V is also a Banach space respect to N, then we get that $\sum_{j=1}^\infty v_j$ converges in V.

Of course, if $\sum_{j=1}^\infty v_j$ converges in V, then the corresponding sequence of partial sums is a Cauchy sequence, so that (2.10.6) holds. If N is an ultranorm on V, then

$$N\Big(\sum_{j=1}^{\infty} v_j\Big) \le \max_{j \ge 1} N(v_j), \tag{2.10.11}$$

because of (2.10.10). More precisely, the maximum on the right is automatically attained when (2.10.6) holds. Indeed, if $v_j \ne 0$ for some j, then the maximum on the right side may be reduced to the maximum of finitely many terms.

2.11 Norms on associative algebras

Let k be a field, and let A be an associative algebra over k. This means that A is a vector space over k, equipped with a binary operation that is bilinear over

k and associative. Sometimes one may also ask that A have a multiplicative identity element $e = e_A$.

Let $|\cdot|$ be an absolute value function on k, and let N be a norm on A with respect to $|\cdot|$ on k. We say that N is *submultiplicative* on A if

$$N(x\,y) \le N(x)\,N(y) \tag{2.11.1}$$

for every $x, y \in A$.

If A has a multiplicative identity element e, then (2.11.1) implies that

$$N(e) \le N(e)^2. \tag{2.11.2}$$

Of course, if $e = 0$, then $A = \{0\}$. Otherwise, (2.11.2) implies that $N(e) \ge 1$. Sometimes one may ask that $N(e) = 1$.

One can use submultiplicativity to get that multiplication is continuous as a mapping from $A \times A$ into A, using a suitable norm on $A \times A$, as in Section 2.3. In fact, the restriction of this mapping to any bounded subset of $A \times A$ is uniformly continuous.

If N is submultiplicative on A, and A is complete with respect to the metric associated to N, then A is said to be a *Banach algebra* over k with respect to N. Otherwise, one can use a completion of A to get a Banach algebra, as usual.

More precisely, one can extend multiplication on A to a completion using the uniform continuity of multiplication on bounded subsets of $A \times A$. If one uses the space of Cauchy sequences in A to get a completion, then one can define multiplication on the completion more directly in terms of products of Cauchy sequences.

Suppose that A has a multiplicative identity element e, and let $x \in A$ be given. Let n be a nonnegative integer, and note that

$$(e - x)\sum_{j=0}^{n} x^j = \Big(\sum_{j=0}^{n} x^j\Big)(e - x) = e - x^{n+1}, \tag{2.11.3}$$

by a standard argument, where x^0 is interpreted as being equal to e, as usual. If $e - x$ has a multiplicative inverse in A, then

$$\sum_{j=0}^{n} x^j = (e - x)^{-1}(e - x^{n+1}) = (e - x^{n+1})(e - x)^{-1}. \tag{2.11.4}$$

If N is submultiplicative on A, then

$$N(x^l) \le N(x)^l \tag{2.11.5}$$

for every positive integer l. If

$$N(x) < 1, \tag{2.11.6}$$

then it follows that $\sum_{j=0}^{\infty} x^j$ converges in A, with

$$\sum_{j=0}^{\infty} x^j = (e - x)^{-1}. \tag{2.11.7}$$

In fact, (2.11.6) implies that $\sum_{j=0}^{\infty} x^j$ converges absolutely with respect to N, because of (2.11.5). If A is a Banach algebra, then it follows that $\sum_{j=0}^{\infty} x^j$ converges in A, as in the previous section. In this case, one can use (2.11.3) to get that $e - x$ has a multiplicative inverse in A, as in (2.11.7).

2.12 More on the archimedean property

Let k be a field with an absolute value function $|\cdot|$, and suppose that $|\cdot|$ is archimedean on k, as in Section 1.5. Note that this implies that k has characteristic 0. This means that there is an embedding of $\mathbf{Q}$ into k, as usual. This leads to an absolute value function on $\mathbf{Q}$, corresponding to $|\cdot|$ on k.

It is easy to see that this absolute value function on $\mathbf{Q}$ is archimedean too. This implies that this absolute value function on $\mathbf{Q}$ is equivalent to the standard absolute value function on $\mathbf{Q}$, as in Section 1.11.

If a is any positive real number, then $|\cdot|^a$ is a quasimetric absolute value function on k, as in Section 1.4. We can choose a so that $|\cdot|^a$ corresponds exactly to the standard absolute value function on $\mathbf{Q}$ with respect to the standard embedding of $\mathbf{Q}$ into k, as in the preceding paragraph. This implies that $|\cdot|^a$ is an absolute value function on k, as in Section 1.6. This shows that $|\cdot|$ is equivalent to an absolute value function on k such that the corresponding absolute value function on $\mathbf{Q}$ is the standard absolute value function.

Suppose now that the absolute value function on $\mathbf{Q}$ induced by $|\cdot|$ on k is the standard absolute value function on k. Suppose also that k is complete with respect to the metric associated to $|\cdot|$ on k. Under these conditions, the standard embedding from $\mathbf{Q}$ into k has a unique extension to an embedding from $\mathbf{R}$ into k such that $|\cdot|$ on k corresponds to the standard absolute value function on $\mathbf{R}$ with respect to this embedding, as in Section 1.15. This corresponds to some remarks after the statement of Theorem 1.1 on p33 of [33], and at the beginning of the proof of Theorem 16 on p260 of [95].

Another famous theorem of Ostrowski states that either this embedding of $\mathbf{R}$ into k is surjective, or otherwise k is isomorphic to $\mathbf{C}$, in such a way that $|\cdot|$ on k corresponds to the standard absolute value function on $\mathbf{C}$. This corresponds to Theorem 1.1 on p33 of [33], and Theorem 16 on p260 of [95]. This also corresponds to the corollary on p291 of [111], where one starts with a field with an absolute value function that contains $\mathbf{R}$, and where the restriction of the absolute value function on the larger field to $\mathbf{R}$ is the standard absolute value function on $\mathbf{R}$. Note that the fact that an archimedean absolute value function on $\mathbf{Q}$ is equivalent to the standard absolute value function on $\mathbf{Q}$ is mentioned on p285 of [111].

Observe that the standard absolute value function on $\mathbf{C}$ is the only absolute value function on $\mathbf{C}$ that agrees with the standard absolute value function on $\mathbf{R}$, as in Section 2.7. This corresponds to Lemma 2.1 on p34 of [33], and to a remark near the top of p261 of [95].

If k is as before and k contains an element whose square is -1, then the previous embedding of $\mathbf{R}$ into k may be extended to an embedding of $\mathbf{C}$ into k.

The remark in the preceding paragraph implies that the absolute value function on k corresponds to the standard absolute value function on $\mathbf{C}$ with respect to this embedding.

In this case, Ostrowski's theorem is related to a well-known result about complex Banach algebras, and this will be discussed further in the next section.

2.13 Algebras over the complex numbers

Let A be an associative algebra over the complex numbers with a nonzero multiplicative identity element e, and let N be a submultiplicative norm on A with respect to the standard absolute value function on $\mathbf{C}$. If $x \in A$, then it is well known that there is a $\lambda \in \mathbf{C}$ such that $\lambda\, e - x$ is not invertible in A. This is normally stated for complex Banach algebras, and one can reduce to that case using a completion of A.

Otherwise, if $\lambda\, e - x$ is invertible in A for every $\lambda \in \mathbf{C}$, then one can show that

$$(\lambda\, e - x)^{-1} \tag{2.13.1}$$

is holomorphic as an A-valued function of λ on the complex plane. In particular, if μ is a continuous linear functional on A, then

$$\mu((\lambda\, e - x)^{-1}) \tag{2.13.2}$$

is a holomorphic complex-valued function of $\lambda \in \mathbf{C}$. One can also show that these functions then to 0 as $|\lambda| \to \infty$. This implies that

$$\mu((\lambda\, e - x)^{-1}) = 0 \tag{2.13.3}$$

for every $\lambda \in \mathbf{C}$, by standard results in complex analysis.

It is well known that the continuous linear functionals on A separate points, because of the Hahn–Banach theorem. Thus (2.13.3) implies that (2.13.1) is equal to 0 for every $\lambda \in \mathbf{C}$. However, (2.13.1) is nonzero for every λ, because $e \neq 0$, by hypothesis.

If every nonzero element of A is invertible, then it follows that every element of A is of the form $\lambda\, e$ for some $\lambda \in \mathbf{C}$. This is a famous theorem of Gelfand and Mazur. This is also normally stated for complex Banach algebras, although completeness does not seem to be needed here.

This implies Ostrowski's theorem in the case where the field contains an element whose square is -1, as in the previous section. This is related to some remarks beginning on p39 of [33]. See also [98, 175, 195].

See also Lemma 3.1 on p38 of [33] and the proof of Theorem 16 on p260 of [95] for Ostrowski's theorem in this case,

Of course, A may be considered as an associative algebra over the real numbers as well. Suppose now that N is a norm on A as a vector space over the real numbers, and that N is submultiplicative on A. If A is commutative, then Theorem 1 on p289 of [111] states that for every $x \in A$ there is a $\lambda \in \mathbf{C}$ such that $\lambda\, e - x$ is not invertible in A, as before. The proof, due to Tornheim, does not use complex analysis.

2.14 The case where there is no square root of -1 in k

To finish the proof of Ostrowski's theorem from Section 2.12, one should consider the case where there is no element of k whose square is equal to -1. One would like to show that the natural embedding from $\mathbf{R}$ into k discussed previously is surjective under these conditions.

To do this, one can adjoin a square root of -1 to k, and try to show that this larger field is equivalent to $\mathbf{C}$. In the proof of the corollary on p290 of [111], a submultiplicative norm is defined on the extension, considered as a commutative algebra over the real numbers. Theorem 1 on p289 of [111] can be used to get that the extension is equivalent to $\mathbf{C}$, so that k is equivalent to $\mathbf{R}$.

Alternatively, Lemma 2 on p259 of [95] states that the absolute value function on k can be extended to an absolute value function on the extension of k mentioned earlier.

Lemma 2.3 on p37 of [33] says that the absolute value function on k can be extended to a quasimetric absolute value function on the extension of k mentioned before, which works here too.

Chapter 3

3.1 Rearrangements

Let π be a one-to-one mapping from $\mathbf{Z}_+$ onto itself. The infinite series

$$\sum_{j=1}^{\infty} v_{\pi(j)} \tag{3.1.1}$$

is said to be a *rearrangement* of $\sum_{j=1}^{\infty} v_j$. If $\sum_{j=1}^{\infty} a_j$ is an infinite series of nonnegative real numbers, then it is well known and not difficult to show that $\sum_{j=1}^{\infty} a_j$ converges if and only if $\sum_{j=1}^{\infty} a_{\pi(j)}$ converges, in which case the two sums are equal.

3.2 Cauchy products

Let A be a ring, and let $\sum_{j=0}^{\infty} a_j$ and $\sum_{l=0}^{\infty} b_l$ be infinite series with terms in A. Put

$$c_n = \sum_{j=0}^{n} a_j \, b_{n-j} \tag{3.2.1}$$

for each nonnegative inetger n. The corresponding series $\sum_{n=0}^{\infty} c_n$ is called the *Cauchy product* of the series $\sum_{j=0}^{\infty} a_j$ and $\sum_{l=0}^{\infty} b_l$. If $a_j = 0$ for all but finitely many j, and $b_l = 0$ for all but finiely many l, then one can check that $c_n = 0$ for all but finitely many n, and that

$$\sum_{n=0}^{\infty} c_n = \Big(\sum_{j=0}^{\infty} a_j\Big) \Big(\sum_{l=0}^{\infty} b_l\Big). \tag{3.2.2}$$

Suppose for the moment that the a_j's and b_l's are nonnegative real numbers. If N is a nonnegative integer, then it is easy to see that

$$\sum_{n=0}^{N} c_n \le \Big(\sum_{j=0}^{N} a_j\Big) \Big(\sum_{l=0}^{N} b_l\Big) \le \sum_{n=0}^{2N} c_n. \tag{3.2.3}$$

If $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$ converge in $\mathbf{R}$, then it follows that $\sum_{n=0}^\infty c_n$ converges too, and that (3.2.2) holds. Conversely, if $\sum_{n=0}^\infty c_n$ converges, $a_j > 0$ for some j, and $b_l > 0$ for some l, then $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$ converge.

Let k be a field with an absolute value function $|\cdot|$, and suppose that A is an associative algebra over k with a submultipicative norm $\|\cdot\|_A$ with respect to $|\cdot|$ on k. Observe that

$$\|c_n\|_A \le \sum_{j=0}^n \|a_j\|_A \, \|b_{n-l}\|_N \tag{3.2.4}$$

for each $n \ge 0$. The right side is the nth term of the Cauchy product of the series $\sum_{j=0}^\infty \|a_j\|_A$ and $\sum_{l=0}^\infty \|b_l\|_A$. If these series converge, then it follows that $\sum_{n=0}^\infty \|c_n\|_A$ converges, with

$$\sum_{n=0}^\infty \|c_n\|_A \le \sum_{n=0}^\infty \Big(\sum_{j=0}^n \|a_j\|_A \, \|b_{n-j}\|_A\Big) = \Big(\sum_{j=0}^\infty \|a_j\|_A\Big)\Big(\sum_{l=0}^\infty \|b_l\|_A\Big). \tag{3.2.5}$$

Suppose that $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$ also converge in A, which holds automatically when A is a Banach algebra with respect to $\|\cdot\|_A$, as in Section 2.10. Under these conditions, $\sum_{n=0}^\infty c_n$ converges in A, with sum as in (3.2.2). Indeed, if N is a nonnegative integer, then

$$\Big(\sum_{j=0}^N a_j\Big)\Big(\sum_{l=0}^N b_l\Big) - \sum_{n=0}^N c_n = \sum_{\substack{j,l \le N \\ j+l>N}} a_j \, b_l, \tag{3.2.6}$$

where more precisely the sum on the right is taken over all nonnegative integers j, l satisfying the indicated conditions. This implies that

$$\Big\|\Big(\sum_{j=0}^N a_j\Big)\Big(\sum_{l=0}^N b_l\Big) - \sum_{n=0}^N c_n\Big\|_A \le \sum_{\substack{j,l \le N \\ j+l>N}} \|a_j\|_A \, \|b_l\|_A. \tag{3.2.7}$$

One can check that the right side tends to 0 as $N \to \infty$ when $\sum_{j=0}^\infty \|a_j\|_A$ and $\sum_{l=0}^\infty \|b_l\|_A$ converge.

If $\|\cdot\|_A$ is an ultranorm on A, then

$$\|c_n\|_A \le \max_{0 \le j \le n} (\|a_j\|_A \, \|b_{n-j}\|_A) \tag{3.2.8}$$

for each $n \ge 0$. Using this, one can check that

$$\|c_j\|_A \to 0 \text{ as } n \to \infty \tag{3.2.9}$$

when

$$\|a_j\|_A \to 0 \text{ as } j \to \infty \text{ and } \|b_l\|_A \to 0 \text{ as } l \to \infty. \tag{3.2.10}$$

We also have that

$$\Big\|\Big(\sum_{j=0}^N a_j\Big)\Big(\sum_{l=0}^N b_l\Big) - \sum_{n=0}^N c_n\Big\|_A \le \max\{\|a_j\|_A\,\|b_l\|_A : j, l \le N,\ j + l > N\} \tag{3.2.11}$$

for every nonnegative integer N in this case. One can verify that the right side tends to 0 as $N \to \infty$ when (3.2.10) holds. This implies that $\sum_{n=0}^\infty c_n$ converges in A, with sum as in (3.2.2), when $\sum_{j=0}^\infty a_j$ and $\sum_{l=0}^\infty b_l$ converge in A.

Chapter 4

Formal polynomials and power series

4.1 Formal polynomials

If n is a positive integer, then a *multi-index* of length n is an n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers. In tis case, we may put

$$|\alpha| = \sum_{j=1}^{n} \alpha_j. \tag{4.1.1}$$

Let A be a ring, and let $T_1, \ldots, T_n$ be n distinct commuting indeterminates, which are also considered to commute with the elements of A. If α is a multi-index, then

$$T^\alpha = T_1^{\alpha_1} \cdots T_n^{\alpha_n} \tag{4.1.2}$$

is the corresponding *formal monomial* in $T_1, \ldots, T_n$. If β is another multi-index, then we put

$$T^\alpha \, T^\beta = T^{\alpha+\beta}. \tag{4.1.3}$$

Chapter 5

5.1 Some remarks about field extensions

Let k be a field, and let k_1 be a field that contains k_1 as a subfield.

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